Quantum Wasserstein isometries of n-qubit state spaces with respect to the symmetric transport cost

Eszter Szabó

Advisor: Dániel Virosztek

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Motivation and background

- classical optimal transport problem: Monge (1781), Kantorovich (1940s)
- quantum optimal transport with quantum channels: de Palma & Trevisan (2021)
- application areas: fluid mechanics, machine learning, image processing and economics

Classical optimal transport I.

- initial distribution on X : $(X, \mathcal{A}_X, \mu)$
- final distribution on Y : $(Y, \mathcal{A}_Y, \nu)$
- cost function $c(x, y) : X \times Y \to \overline{\mathbb{R}}_+$

Definition

A transport plan between the probability measures $\mu \in \mathrm{Prob}(X)$ and $\nu \in \mathrm{Prob}(Y)$ is a probability measure $\pi \in \mathrm{Prob}(X \times Y)$ such that

$$\int_Y d\pi(x, y) = d\mu(x) \text{ and } \int_X d\pi(x, y) = d\nu(y).$$

Classical optimal transport II.

- $\Pi(\mu,\nu)$ denotes the set of transport plans from μ to ν
- goal: minimizing the transport cost, i.e. finding

$$\inf_{\pi\in\Pi(\mu,\nu)} I[\pi] = \inf_{\pi\in\Pi(\mu,\nu)} \left(\int_{X\times Y} c(x,y)d\pi(x,y) \right).$$

Wasserstein distance

- if $X = Y$ and the cost function has the form $c(x,y) := d(x,y)^p$ where $d(x,y)^p$ is a metric
- a metric in the space of probability measures on X is:

Definition

Wasserstein distance of order p:

$$W_p(\mu,\nu) := \inf_{\pi\in\Pi(\mu,\nu)} \left(\int_{X\times X} d(x,y)^p d\pi(x,y)\right)^{1/p}.$$

Quantum optimal transport I.

- underlying Hilbert space : $\mathcal{H} = \mathbb{C}^{2^n}$
- Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Quantum optimal transport II. (Quantum states)

- set of quantum states in $\mathcal{H} = \mathbb{C}^{2^n}$:

$$\mathcal{S}\left(\mathbb{C}^{2^n}\right) = \left\{\varrho \in \mathcal{L}\left(\mathbb{C}^{2^n}\right) : \varrho \geq 0, \mathrm{tr}[\varrho] = 1\right\}.$$

- pure state: rank 1 projection, i.e. there exists a unit vector $\psi \in \mathbb{C}^{2^n}$ such that $\varrho = |\psi\rangle\langle\psi|$.
- the set of pure states: $\mathcal{P}_1\left(\mathbb{C}^{2^n}\right)$

Quantum optimal transport III.

- The set of couplings of states $\varrho, \omega \in \mathcal{S}(\mathcal{H})$:

$$C(\varrho,\omega) = \left\{ \Pi \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H}^*) : \mathrm{tr}_{\mathcal{H}^*}[\Pi] = \omega, \mathrm{tr}_{\mathcal{H}}[\Pi] = \varrho^T \right\},$$

  where $\mathrm{tr}_{\mathcal{H}^*}[\Pi]$ and $\mathrm{tr}_{\mathcal{H}}[\Pi]$ stand for the partial traces of Π.
- The quantum Wasserstein distance of states ϱ and ω with respect to a cost operator C:

$$D_C(\varrho,\omega) = \left(\inf_{\Pi \in C(\varrho,\omega)} \mathrm{tr}_{\mathcal{H}\otimes\mathcal{H}^*}[\Pi C] \right)^{1/2}.$$

Quantum optimal transport IV. (Quantum Wasserstein isometries)

Definition

A map $\Phi : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ is an isometry of the quantum Wasserstein distance D_C if

$$D_C(\Phi(\varrho), \Phi(\omega)) = D_C(\varrho, \omega)$$

holds for all states $\varrho, \omega \in \mathcal{S}(\mathcal{H})$.

New results I. (Symmetric cost in 2^n dimension)

Definition

$$C_{\text{sym}}^{(n)} = \sum_{j_1,\ldots,j_n=0}^{3} \left((\sigma_{j_1} \otimes \ldots \otimes \sigma_{j_n}) \otimes I_{2^n}^T - I_{2^n} \otimes (\sigma_{j_1} \otimes \ldots \otimes \sigma_{j_n})^T \right)^2 .$$

Theorem

$$C_{sym}^{(n)} = 2 \cdot 4^n \cdot I_{2^n} \otimes I_{2^n}^T - 2 \cdot 4^n \cdot \frac{1}{2^n} \left|\left| I_{2^n} \right\rangle\right\rangle \left\langle\left\langle I_{2^n} \right|\right| .$$

New results II. (corollaries of the spectral decomposition)

Corollary

$$C_{sym}^{(n)} \leq 2 \cdot 4^n \cdot I_{2^n} \otimes I_{2^n}^T,$$

where $\leq$ stands for the Loewner order (positive semidefinite order).

by estimating the distance of two states $\varrho, \omega \in \mathcal{S}\left(\mathbb{C}^{2^n}\right)$ from above with the cost respect to the trivial coupling $\omega \otimes \varrho^T$ we obtain

Corollary

$$\left(D_{sym}^{(n)}(\varrho, \omega)\right)^2 \leq 2 \cdot 4^n \quad \forall \varrho, \omega \in \mathcal{S}\left(\mathbb{C}^{2^n}\right).$$

New results III. (corollaries of the spectral decomposition)

- by explicit calculation of the transport cost for the trivial coupling $\omega \otimes \varrho^T$:

$$\mathrm{tr}\left[C_{\mathrm{sym}}^{(n)}(\omega \otimes \varrho^T)\right] = 2 \cdot 4^n \left(1 - \frac{1}{2^n}\mathrm{tr}\left[\omega\varrho\right]\right),$$

Corollary

$$\left(D_{sym}^{(n)}(\varrho, \omega)\right)^2 = 2 \cdot 4^n \Longrightarrow tr[\omega\varrho] = 0.$$

- If at least one of the states is pure, the only coupling is the trivial, so the converse implication is also true.

New results IV.

Definition

Diameter of a qubit state space:

$$\operatorname{diam}\left(\mathcal{S}\left(\mathbb{C}^{2^n}\right), D_{\mathrm{sym}}\right) = \sup_{\varrho,\omega \in \mathcal{S}\left(\mathbb{C}^{2^n}\right)} D_{\mathrm{sym}}(\varrho, \omega).$$

Our following result states that a state ϱ is pure if and only if there exists a 2^n-simplex in the space of states with maximal quantum Wasserstein distance between all its vertices and one of its vertices being ϱ.

New results V.

Theorem

For any state $\varrho \in \mathcal{S}(\mathbb{C}^{2^n})$ *the following are equivalent:*

1. $\varrho \in \mathcal{P}_1(\mathbb{C}^{2^n})$, *i.e.* ϱ *is pure*
2. $\exists \varrho_1 \varrho_2, \dots, \varrho_{2^n-1} \in \mathcal{S}(\mathbb{C}^{2^n})$ *such that*

$$D_{sym}(\varrho_j, \varrho_k) = diam\left(\mathcal{S}(\mathbb{C}^{2^n}), D_{sym}\right)$$

for any $j \neq k \in \{0, 1, \dots, 2^n - 1\}$ *with* $\varrho_0 = \varrho$.

Corollary

If $\Phi : \mathcal{S}(\mathbb{C}^{2^n}) \to \mathcal{S}(\mathbb{C}^{2^n})$ *is a* D_{sym}*-isometry, then* Φ *maps pure states to pure states.*

New results VI. (Symmetry transformations)

Definition

An operator $T : \mathcal{S}(\mathbb{C}^{2^n}) \to \mathcal{S}(\mathbb{C}^{2^n})$ is a symmetry transformation if it preserves transition probabilities, i.e.

$$\operatorname{tr}[T(\varrho)T(\omega)] = \operatorname{tr}[\varrho\omega] \quad \text{for all } \varrho, \omega \in \mathcal{S}(\mathbb{C}^{2^n}).$$

Theorem

Wigner theorem:
If $T : \mathcal{S}(\mathbb{C}^{2^n}) \to \mathcal{S}(\mathbb{C}^{2^n})$ is a symmetry transformation, then there exists a unitary or anti-unitary operator U such that

$$T(\varrho) = U\varrho U^\dagger \quad \textit{for all } \varrho \in \mathcal{S}(\mathbb{C}^{2^n}).$$

New results VII: (A Wigner-type characterization)

The following theorem gives a characterisation of D_{sym}-isometries in $\mathcal{S}(\mathbb{C}^{2^n})$.

Theorem

Let $\Phi : \mathcal{S}(\mathbb{C}^{2^n}) \to \mathcal{S}(\mathbb{C}^{2^n})$ *be a map.*
The following are equivalent:

1. Φ *is a* D_{sym}*-isometry*
2. *there exists a unitary or anti-unitary operator* U *such that*

$$\Phi(\varrho) = U\varrho U^{\dagger} \quad \text{for all } \varrho \in \mathcal{S}(\mathbb{C}^{2^n}).$$

Future goals

- removing one Pauli matrix from the set of generators makes the structure of quantum Wasserstein isometries dramatically different in the single qubit space $\rightsquigarrow$ cost operators generated by a subset of all tensor products of Pauli matrices in n-qubit systems
- quantum Wasserstein isometries on systems where the dimension of the underlying Hilbert space is not a power of two
- infinite-dimensional generalizations of our n-qubit results

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Thank you for your attention!