Introduction to Differential Topology

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Motivation and Introduction

During my studies in pure mathematics, I have become increasingly interested in the geometric side of mathematics. Although I had some preliminary exposure to manifolds and smooth maps, I realised early in the semester that my background in differential topology was rather limited. As I intend to pursue my academic future in geometry, a solid understanding of the differential structure of manifolds feels indispensable.

For this reason, in the Directed studies course this semester I have been studying differential topology using the book Differential Topology by Victor Guillemin and Alan Pollack as a primary reference, focusing on the material from the the first chapter through approximately the middle of the second chapter, and solving as many exercises as possible in order to gain a deeper understanding of the fundamental tools of differential topology.

Abstract

This report summarises an introductory study of differential topology, focusing on the theory of smooth manifolds, smooth maps, transversality and intersections. After developing a basic theory of manifolds and tangent spaces, the discussion is extended to manifolds with boundary. The final part of the report outlines transversality theorems and their applications to intersection theory, leading to the definition of the mod 2 intersection number.

1 MANIFOLDS AND SMOOTH MAPS

We are very familiar with Euclidean spaces however, many topological spaces look quite different from them when viewed as a whole. Yet, if we focus on sufficiently small neighbourhoods of a point, these spaces can behave just like $\mathbb{R}^n$. This simple idea leads to the notion of *smooth manifold* and forms a starting point of differential topology.

Those topological manifolds that have a differentiable structure — that is, those that are locally diffeomorphic to $\mathbb{R}^n$ — become smooth manifolds. In topology we consider spaces only up to homeomorphism, so continuous deformations do not change the nature of a space. This flexible viewpoint however, becomes more rigid once we introduce analysis. For example, although a cone is a topological manifold, it fails to be a smooth manifold since its vertex does not admit a neighbourhood that is smoothly compatible with $\mathbb{R}^n$.

Formally, X is a k-dimensional smooth manifold if for every point $x \in X$ there exists a neighborhood V of x and an open set $U \subset \mathbb{R}^k$ such that the map $\phi\colon U \to V$ is a diffeomorphism. Such a map is called a *local parametrisation*, and its inverse is called a *local coordinate system* or *chart*. These maps play a fundamental role in understanding smooth manifolds. Although differentiation is not defined intrinsically on a smooth manifold, charts allow us to transfer the problem to $\mathbb{R}^k$ where the usual tools of analysis are available. Furthermore, smooth manifolds can be constructed by choosing a family of charts, known as *atlas*, whose transition maps are smooth.

To gain a clearer picture of the local structure of a smooth manifold, we make use of tangent spaces. At each point x of the manifold X, the tangent space T_xX encodes the tangent directions determined by smooth curves passing through x, providing the best linear approximation to the manifold at that point. It can be shown that the tangent space at x is equal to the image of the differential of the parametrisation, i.e. $T_xX = \mathrm{Im}(d\phi_0)$ where $\phi(0) = x$. If X and Y are smooth manifolds and $f\colon X \to Y$ is a smooth map, then its differential $df_x\colon T_xX \to T_{f(x)}Y$ is a linear transformation between the tangent spaces. Since we understand linear transformations well, studying df_x allows us to classify and analyze smooth maps between manifolds in terms of their local behaviour. There are three possible situations regarding the dimensions of the manifolds X and Y:

(i) $\dim X = \dim Y$: This is the most favourable case, where the differential df_x is a linear isomorphism, and thus by the Inverse Function Theorem f is a local diffeomorphism at x.
(ii) $\dim X < \dim Y$: In this case, the best scenario occurs when df_x is injective, and then f is called immersion at x.
(iii) $\dim X > \dim Y$: Here, the ideal situation is that df_x is surjective, and then f is called submersion at x.

Being an immersion is a local property. To make stronger statements about the map f on a global scale, we need to impose additional global conditions, such as properness, injectivity, or compactness. This distinction is necessary because there exist immersions whose images are not submanifolds. To resolve this issue, the notion of proper maps will be introduced and used to define embeddings. An embedding is an immersion that is both injective and proper. For such a map, the global image is guaranteed to be a smooth submanifold of the target manifold.

A similar issue arises for submersions. For a smooth map $f\colon X \to Y$, the preimage $f^{-1}(y)$ of a point $y \in Y$ is not necessarily a smooth manifold. This problem is resolved by the Preimage Theorem, which states that if y is a regular value of f, then $f^{-1}(y)$ is indeed a smooth submanifold of X. As a result, the Preimage Theorem provides a much simpler method for constructing submanifolds than using local parametrisations.

Going one step further, one can study the preimage of an entire submanifold — not just a single point — using the notion of transversality. The corresponding theorem states that if $f\colon X \to Y$ is a smooth map and $Z \subset X$ is a submanifold, then whenever f is transverse to Z, the preimage $f^{-1}(Z)$ is a submanifold of X. In another approach, one can study the transversality of two submanifolds. If Z and W are submanifolds of a manifold Y and they are transversal, then their intersection also provides a submanifold of Y.

2 MANIFOLDS WITH BOUNDARY

Extending the notion of a manifold by allowing boundaries significantly enlarges the class of geometric objects that one studies. In this case, one appropriate local model for a manifold with boundary is the upper half-space $H^k = \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_k \geq 0\}$, whose boundary is the hyperplane R^{k-1}, i.e. $\partial H^k = \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_k = 0\}$ and $\text{int}(H^k) = \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_k > 0\}$. Therefore, a subset $X \subset \mathbb{R}^k$ is called *k-dimesional manifold with boundary* if for every point $x \in X$ there exists a neighbourhood $V \subset X$, an open set $U \subset H^k$ and a diffeomorphism $\phi\colon U \to V$. The boundary of X is defined as $\partial X = \{x \in X \mid x = \phi(z) \text{ for some } z \in \text{int}(H^k) \text{ and } \phi \text{ is a local parametrisation}\}$ and $\dim \partial X = k - 1$. The interior of X is $\text{int}(X) = X - \partial X$, which is a k-dimensional manifold just like X. Thus, the smooth manifolds we have discussed in the previous section can be viewed as special cases of manifolds with boundary, where the boundary happens to be empty.

Manifolds with boundary are not singular spaces; in particular, one can assign a tangent space to every point, including points on the boundary. Boundary points are not singular because they possess neighbourhoods modelled on H^k, set up with the subspace topology inherited from R^k. This allows us to define local parametrisations and tangent spaces at boundary points in the same systematic way as for interior points. The product of two manifolds with boundary is not necessarily a manifold with boundary. For example, consider the square $[0, 1] \times [0, 1]$. Although each factor is a 1-dimensional manifold with boundary, their product has four singular points at the corners. However, it is true that the product of a smooth manifold (without boundary) and a manifold with boundary is a manifold with boundary.

The question rises: under what conditions on a smooth map $f\colon X \to Y$, can we ensure that $f^{-1}(Z)$ is a manifold with boundary, where X, Y are manifolds with $\partial X \neq \emptyset, \partial Y = \emptyset$ and $Z \subset Y$ is a submanifold? The answer is provided by the transversality theorem for manifolds with boundary. In this setting, two conditions are required: f must be transversal to Z and the $\partial f = f|_{\partial X}\colon \partial X \to Y$ must also be transversal to Z. When both conditions hold, the preimage $f^{-1}(Z)$ is a manifold with boundary and $\partial f^{-1}(Z) = f^{-1}(Z) \cap \partial X$. This result generalises the preimage theorem in the boundary-free case.

3 TRANSVERSALITY AND INTERSECTION

Given an arbitrary smooth map $f\colon X \to Y$ is not necessarily transversal to a given submanifold $Z \subset Y$. However, f can be deformed by an arbitrarily small amount into a map that is transversal to Z. The key idea behind this result is the use of families of mappings.

Suppose $f_s\colon X \to Y$ is a family of smooth maps indexed by a parameter s ranging over a set S. As in the case of homotopies, we consider the associated map $F\colon X \times S \to Y$, $F(x, s) = f_s(x)$. To ensure that the family varies smoothly, one can assume that S is a smooth manifold and that the map F is smooth. With this setup, we arrive to the Transversality Theorem. Let $F\colon X \times S \to Y$ be a smooth map of manifolds, where only X have a boundary, and let $Z \subset Y$ be a submanifold without boundary. If both F and ∂F are transversal to Z, then for almost every $s \in S$, both f_s and ∂f_s are transversal to Z.

The Transversality Homotopy Theorem states that for any smooth map $f\colon X \to Y$ and for any boundaryless submanifold Z of the boundaryless manifold Y, there exists a smooth map $g\colon X \to Y$ homotopic to f such that both g and ∂g are transversal to Z.

Finally, there is to be mentioned the Extension Theorem, which states that if $f\colon X \to Y$ is smooth, $Z \subset Y$, $\partial Z = \emptyset$, $\partial Y = \emptyset$ and C is a closed subset of X such that f is transversal to Z on C. Then there exists a smooth map $g \colon X \to Y$ homotopic to f such that g and ∂g are transversal to Z and on a neighbourhood of C $f = g$.

These theorems and their proofs are subtle and technically demanding. Nevertheless, the effort invested in understanding them is well rewarded. Using these preceding results, one can study a simple and intuitive homotopic invariant for intersection manifolds.

Let X and Z be submanifolds of Y such that $\dim(X) + \dim(Z) = \dim(Y)$. If X and Y intersect transversely, then $\dim(X \cap Y) = 0$. Furthermore, if X and Z are closed and at least one of them is compact, then their intersection consists of finitely many points. The cardinality of this set is defined as the *intersection number* of X and Z and is denoted by $\#(X \cap Z)$. A problem arises from the fact that the intersection number depends on the deformation of X. Fortunately, this problem can be solved since the intersection number obtained with different deformations agree on mod 2.

Defining the *mod 2 intersection number* of the map $f \colon X \to Y$ with $I_2(f, Z)$ as the number of points in f^{-1} mod 2 leads to the theorem, that states if $f_0, f_1 \colon X \to Y$ are homotopic and both transversal to Z, then $I_2(f_0, Z) = I_2(f_1, Z)$.

If $I_2(X, Z) \neq 0$ then there exists no deformation of X with which one could separate X from Z, in terms of intersections. If $X = \partial W$ where $W \subset Y$ then $I_2(X, Z) = 0$. If X and Z are transversal then intuitively this means that Z must pass out of W as often as it enters, hence $\#(X \cap Z)$ is even.

Intersection theory gives an interesting homotopy invariant attached to maps between manifolds of the same dimension. If $f \colon X \to Y$ is a smooth map of a compact manifold X into a connected manifold Y and $\dim(X) = \dim(Y)$, then $I_2(f, \{y\})$ is the same for all points $y \in Y$. This common value is called *mod 2 degree of* f and denoted $\deg_2(f)$.

Since $\deg_2(f)$ is an intersection number then it is a homotopy invariant, i.e. if f_0 and f_1 are homotopic maps then $\deg_2(f_0) = \deg_2(f_1)$. Another consequence would be that if $X = \partial W$ for some compact manifold W with boundary, and if $f \colon X \to Y$ can be extended to the entire W, then $\deg_2(f) = 0$.