Cosmological polytopes

Individual project 1

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Motivation

- *Cosmological polytopes* arise in the study of Feynman diagrams
- Their geometry encodes physical information
- Computing physical quantities requires *triangulating* them
- *Root polytopes* are often easier to triangulate

Goal: Use ideas from root polytopes to understand triangulations of cosmological polytopes.

The cosmological polytope and the root polytope

Let $G = (V, E)$ be a connected, undirected graph.

- Work in $\mathbb{R}^{|V|+|E|}$ with basis vectors: $\{x_i : i \in V\}$, $\{x_e : e \in E\}$

Definition

The *cosmological polytope* of $G = (V, E)$ is
$C_G = \text{conv}\{x_i + x_j - x_e,\ x_i - x_j + x_e,\ -x_i + x_j + x_e \ :\ e = ij \in E\}$.
This is a polytope of dimension $|V| + |E| - 1$.

Definition

Let G be a bipartite graph with color classes V and E. The *root polytope* of G is
$Q_G = \text{conv}\{x_v + x_e, :\ v \in V,\ e \in E,\ ve \text{ is an edge in } G\}$.
This is a polytope of dimension $|V| + |E| - 2$.

Basic question

Question: Can C_G be realized as a root polytope?

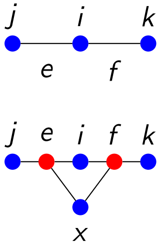
- C_G and Q_G have different dimensions ($\dim(C_G) = |V| + |E| - 1$ and $\dim(Q_G) = |V| - 1$)
- They cannot coincide for the same graph

Question: Do there exist a graph G' such that $C_G = Q_{G'}$?

Example: $C_G = Q_{G'}$

Let G be a path with vertices i, j, k.

- C_G has 6 vertices in $\mathbb{R}^5$
- 4 simplices, 2 possible triangulations



Idea: The vertices of C_G can be viewed as a vectormatroid, and it has a graphic realization.

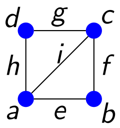
- The realization is a bipartite graph (G')
- $Q_{G'}$ has the same vertex dependencies

Result: Identify vertices appropriately $\Rightarrow C_G \cong Q_{G'}$ and their triangulations match under this correspondence

General construction

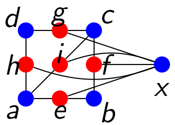
For a given graph G:

- Subdivide each edge by a new vertex
- Add a new vertex x
- Connect x to all "edge-vertices"



We obtain a bipartite graph G' :

- $|V(C_G)| = |V(Q_{G'})|$
- $\dim C_G = \dim Q_{G'}$



There is a natural bijection between the vertices of the two polytopes

Preservation of dependencies

Proposition

The construction preserves linear dependencies among vertices.

Limitations:
Although the bijection preserves linear dependencies,but it does not preserve triangulations in general.

Open problem:

Can the construction be modified to preserve triangulations?

Thank you for your attention!