IGUSA LOCAL ZETA FUNCTION AND FRACTAL STRINGS

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1. Introduction

A *fractal string* (Ω) on the real line is a bounded open set, and as such it is the disjoint union of countably many open intervals. Thus to a fractal string we may assign to series $\mathcal{L} = l_1, l_2, l_3, \ldots$, where the l_j are the lengths of these intervals. We may assume that there is an infinite number of them and that we listed them in a monotone decreasing way. Then the geometric fractal zeta function of the fractal string can be defined as

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} l_j^s,$$

the fractal zeta function's abscissa of convergence as

$$\sigma_{\mathcal{L}} = \inf\{\alpha \in \mathbb{R} : \sum_{j=1}^{\infty} l_j^s \text{ converges if } \operatorname{Re} s > \alpha\},$$

and the fractal string's Minkowski dimension as

$$D_{\mathcal{L}} = \inf\{\alpha \geq 0 : \mathrm{vol}(\varepsilon) = O(\varepsilon^{1-\alpha}) \text{ as } \varepsilon \to 0^+\},$$

where $\mathrm{vol}(\varepsilon) = \mathrm{vol}_1\{x \in \Omega : d(x, \partial\Omega) < \varepsilon\}$ is the volume of the ε-tube around the boundary of the fractal string.

Lapidus and Van Frankenhuijsen observed and proved that these two values coincide for every fractal string:

Theorem 1.1.

$$\forall \mathcal{L} : D_{\mathcal{L}} = \sigma_{\mathcal{L}}$$

These concepts can be generalized to the p-adic integers.

Definition 1.2. $V \subset \mathbb{Z}_p^n$ is a p-adic fractal string, if it is the disjoint union of countably many balls.

Due to the behaviour of balls in $\mathbb{Z}_p^n$, the decomposition of the p-adic fractal string is not unique, but we may assign to it the maximal ball decomposition and define the p-adic fractal zeta function.

Definition 1.3. Assume that $V = \bigcup_j^* \bigcup_{i=1}^{*k_j} B_{p^{-j}}(x_{ji})$ is a decomposition into maximal balls as above. Then let

$$\zeta_V(s) = \sum_{j=1}^{\infty} k_j p^{-js}$$

the fractal zeta function of the p-adic fractal string V.

The question naturally arises: does Theorem 1.1 still hold for p-adic fractal strings? We have proved that for a special case it stands, the special case being the p-adic fractal strings associated with non-singular affine varieties.

2. p-adic fractal strings associated to affine varieties

Let $\mathbf{f} : \mathbb{Z}_p^n \to \mathbb{Z}_p^m$ be such that $\mathbf{f} = (f_1, \dots, f_m)$, where $f_j \in \mathbb{Z}_p[x_1, \dots, x_n]$ $(n \geq m)$.

Definition 2.1. The affine variety of $\mathbf{f}$ is

$$V(\mathbf{f}) = \{x \in \mathbb{Z}_p^n : \mathbf{f}(x) = 0\}.$$

Definition 2.2. Let us denote the set of so called 'liftable' points of the variety mod p^k by

$$\mathcal{L}_{p^k} = \{\hat{x} \in \mathbb{Z}/p^k\mathbb{Z} | \exists x \in V(\mathbf{f}), [x]_k = \hat{x}\},$$

and their cardinality by

$$L(p^k) = |\mathcal{L}_{p^k}|,$$

where $[x]_k = \sum_{l=0}^{k-1} a_l p^l$, when $x = \sum_{l=0}^{\infty} a_l p^l$.

Theorem 2.3. *When $V(\mathbf{f}) \neq \emptyset$, $\mathbb{Z}_p^n \setminus V(\mathbf{f})$ is a p-adic fractal string. Moreover the p-adic fractal zeta function corresponding to the affine variety is*

$$\zeta_{V(\mathbf{f})}(s) = \sum_{k=1}^{\infty} (p^n L(p^{k-1}) - L(p^k)) p^{-ks}.$$

Proof. If $[x]_k$ can no longer be lifted (where as $[x]_{k-1}$ could be) then the p^{-k} ball around $[x]_k$ is in the complement of $V(\mathbf{f})$, because any point of $\mathbb{Z}_p^n$ can only differ from $[x]_k$ till the kth digit in their formal series representation. The complement is the disjoint union of such balls, thus, $\mathbb{Z}_p^n \setminus V(\mathbf{f})$ is a p-adic fractal string.

By the above reasoning, one can deduce that the number of maximal balls with radius p^{-k} in the complement is

$$|\{x \in \mathbb{Z}_p^n | [x]_k \notin \mathcal{L}_{p^k}, [x]_{k-1} \in \mathcal{L}_{p^{k-1}}\}| = p^n L(p^{k-1}) - L(p^k),$$

proving the formula for $\zeta_{V(\mathbf{f})}(s)$. □

Definition 2.4. We say that $x \in V(\mathbf{f})$ is a non-singular point of the variety $V(\mathbf{f})$, if the Jacobian matrix in x has maximal rank over $\mathbb{Z}/p\mathbb{Z}$. The variety is non-singular if its every point is non-singular.

Definition 2.5. Let us denote the number of liftable non-singular points $\mod p^k$ by

$$N_0(p^k) = |\{\hat{x} \in \mathbb{Z}/p^k\mathbb{Z} | \exists x \in V(\mathbf{f}) \text{ non-singular over } \mathbb{Z}/p\mathbb{Z}, [x]_k = \hat{x}\}|$$

and the number of solutions $\mod p^k$ by

$$N(p^k) = |\{x \in (\mathbb{Z}/p^k\mathbb{Z})^n : \mathbf{f}(x) \equiv 0 \mod p^k\}|.$$

Lemma 2.6. *Let $V_0(\mathbf{f})$ be the set of non-singular points of $V(\mathbf{f})$ that have good reduction. If $V_0(\mathbf{f})$ is non-empty, then the number of solutions of $V_0(\mathbf{f})$ $\mod p^k$ is the following:*

$$N_0(p^k) = p^{(n-m)(k-1)} N_0(p),$$

where .

By Hensel's lemma, all non-singular points are liftable, thus $N_0(p^k)$ is just the number of non-singular points $\mod p^k$. Since all liftable points provide a solution mod p^k, we have the following inequalities

$$\forall k : N_0(p^k) \leq L(p^k) \leq N(p^k).$$

The inequalities might be strict.

Example 2.7. Let $f(x,y) = xy$.

Then $V(f)$ has one singular point: $(0,0)$. Every other point is non-singular.

$$N_0(p^2) = 2p^2 - 2$$
$$L(p^2) = 2p^2 - 1$$
$$N(p^2) = 2p^2$$

since (p,p) is a solution $\mod p^2$, but it cannot be lifted.

2.1. **Non-singular varieties.** By Hensel's lemma, when the variety is non-singular, all points of the variety are liftable, and so

$$\forall k : N_0(p^k) = L(p^k) = N(p^k).$$

Using the recursive formula given by Lemma 2.6, one can easily obtain that in this case, the abscissa of convergence of the p-adic fractal zeta function, defined as

$$\sigma_{V(\mathbf{f})} = \inf\{\alpha \in \mathbb{R} : \sum_{j=1}^{\infty} k_j p^{-js} \text{converges if} \operatorname{Re} s > \alpha\},$$

is $n - m$, the algebraic dimension of $V(\mathbf{f})$.

Similarly, it can be proved that the Minkowski dimension

$$D_{V(\mathbf{f})} = \inf\{\alpha \geq 0 : \operatorname{vol}(\varepsilon) = O(\varepsilon^{n-\alpha}) \text{ as } \varepsilon \to 0^+\},$$

where

$$\operatorname{vol}(\varepsilon) = \lambda(\{x \in \mathbb{Z}_p^n \backslash V(\mathbf{f}) | \exists y \in V(\mathbf{f}) : |x - y|_p < \varepsilon\}),$$

also agrees with $n - m$.

And so, non-singular varieties inherited the property that the Minkowski dimension of the given fractal string coincides with abscissa of convergence of the p-adic fractal zeta function.

2.2. **Singular varieties.** The case of singular varieties is slightly more complicated since, as seen in Example 2.7, our three values may differ from each other.

What we have proved for non-singular varieties will still hold for $\zeta_{V_0(\mathbf{f})}$, assuming $V_0(\mathbf{f})$ is non-empty. Thus, the abscissa of convergence of $\zeta_{V_0(\mathbf{f})}$ is still $n - m$.

Let's denote the series given by $N(p^k)$ as

$$\mathcal{P}_{V(\mathbf{f})}(s) = \sum_{k=1}^{\infty} (p^n N(p^{k-1}) - N(p^k)) p^{-ks}.$$

We have observed that for hypersurfaces this series can be expressed by Igusa's local zeta function:

$$\mathcal{P}_{V(f)}(s) = p^{-(n-s)} Z_{V(f)}(-(n-s)) = p^{-(n-s)} \int_{\mathbb{Z}_p^n} |f(x)|_p^{-(n-s)} d\lambda(x).$$

The abscissa of convergence of $\zeta_{V(\mathbf{f})}$ is between $n-m$ and the abscissa of convergence of $\mathcal{P}_{V(f)}$, but even for hypersurfaces these may not agree.

Example 2.8. Let $f(x,y) = x^2 + y^3$, so $n = 2$, therefore the dimension is 1. Using the stationary phase formula one can calculate that

$$Z_{V(f)}(s) = \frac{1 - p^{-1}}{(1 - p^{-1-s})(1 - p^{-5-6s})}(1 - p^{-2-s} - p^{-2-2s} - p^{-5-6s}).$$

For further information see [3].

So the abscissa of convergence of $p^{s-2}Z_{V(f)}(s-2) = \mathcal{P}_{V(f)}(s)$ is $7/6 > 1$, the dimension of $V(f)$.

Still the abscissa of convergence of $\zeta_{V(f)}(s)$ is equal to 1.

For this f the variety can be parametrized: $V(f) = \{(t^3, -t^2) : t \in \mathbb{Z}_p\}$. Thus

$$L(p^k) = |\{(t^3, -t^2) \in \left(\mathbb{Z}/p^k\mathbb{Z}\right)^2\}|.$$

Observe that $p^{k-1}(p-1) \le L(p^k) \le p^k$, since each unit in $\mathbb{Z}/p^k\mathbb{Z}$ gives rise to a different pair. Now the abscissa of convergence is given by

$$\begin{aligned}\sigma_{V(f)} &= \limsup_{k\to\infty} \frac{\log|\sum_{j=1}^{k} p^2 L(p^{j-1}) - L(p^j)|}{k\log p}\\ &= \limsup_{k\to\infty} \frac{\log|p^2 - L(p^k) + (p^2-1)\sum_{j=1}^{k-1} L(p^j)|}{k\log p}.\end{aligned}$$

Due to the bounds given above, we have:

$$\begin{aligned}(p^2-3)p^k &\le p^2 - p^k + p(p^2-1)(p^{k-1}-1) \le\\ &\le |p^2 - L(p^k) + (p^2-1)\sum_{j=1}^{k-1} L(p^j)| \le p^2 + p^k + p(p+1)(p^{k-2}-1) \le 4p^k\end{aligned}$$

And so, by the fact that the logarithmic function is monotone increasing and the Squeeze theorem, the abscissa of convergence of the p-adic fractal zeta function of the variety is 1, which coincides with the dimension.

One can also get the abscissa of convergence via calculating the values of $L(p^k)$:

Assume $p \neq 2$. Let $t = up^l$, with $u \in \mathbb{Z}_p^\times$. These are points of the form

$$S_l = \{(u^2p^{2l}, u^3p^{3l}) \mod p^k\}.$$

We distinguish 3 cases:

(1) $3l < k$. To count S_l, note that it has the same cardinality as

$$S'_l = \{(u^2, u^3p^l) \mod p^{k-2l}\}$$

Now assume that up^l, vp^l both lead to the same point. Then

$$v \equiv \pm u \mod p^{k-2l}.$$

If $v \equiv -u$, then $v^3 \equiv -u^3 \not\equiv u^3 \mod p^{k-3l}$, since $k - 3l > 0$. Therefore the number of different points arising is $(p-1)p^{k-2l-1}$.

(2) $2l < k \le 3l$. If $p \neq 2$ then the set of points arising from these is

$$\{(u^2p^{2l}, 0) \mod p^k\}.$$

Their number is $|\{u^2 \mod p^{k-2l}\}| = \frac{(p-1)p^{k-2l-1}}{2}$.

(3) $k \le 2l$. There is only one point arising from these: $(0,0) \mod p^k$

To summarize in total this gives

$$1+(p-1)\sum_{l=0}^{\lceil k/3\rceil-1} p^{k-2l-1} + \frac{p-1}{2}\sum_{\lceil k/3\rceil}^{\lceil k/2\rceil-1} p^{k-2l-1} = 1+p^k-p^{k-2\lceil k/3\rceil+1}+\frac{1}{2}(p^{k-2\lceil k/3\rceil}-p^{k-2\lceil k/2\rceil+1})$$

and so

$$L(p^k) = \begin{cases} 1+p^k-p^{k/3+1}+\frac{1}{2}(p^{k/3}-p) & \text{if } k\equiv 0 \mod 6 \\ 1+p^k-p^{(k-1)/3}+\frac{1}{2}(p^{(k+2)/3}-1) & \text{if } k\equiv 1 \mod 6 \\ 1+p^k-p^{(k+1)/3}+\frac{1}{2}(p^{(k+4)/3}-p) & \text{if } k\equiv 2 \mod 6 \\ 1+p^k-p^{k/3+1}+\frac{1}{2}(p^{k/3}-1) & \text{if } k\equiv 3 \mod 6 \\ 1+p^k-p^{(k-1)/3}+\frac{1}{2}(p^{(k+2)/3}-p) & \text{if } k\equiv 4 \mod 6 \\ 1+p^k-p^{(k+1)/3}+\frac{1}{2}(p^{(k+4)/3}-1) & \text{if } k\equiv 5 \mod 6 \end{cases},$$

which again proves that the abscissa of convergence agrees with the dimension. Furthermore, it shows that $\zeta_{V(f)}(s)$ is the sum of finitely many rational functions, hence a rational function in p^{-s}.

Remark 2.9. Note, that the argument using the bounds on $L(p^k)$ works for every $f \in \mathbb{Z}_p[x_1,x_2]$ with variety that can be parametrized as $V(f) = \{(t^a,t^b) : t \in \mathbb{Z}_p\}$, where $\gcd(\gcd(a,b),p(p-1)) = 1$, since in this case the map $\varphi : (\mathbb{Z}/p^k\mathbb{Z})^\times \to (\mathbb{Z}/p^k\mathbb{Z})^\times \times (\mathbb{Z}/p^k\mathbb{Z})^\times, \varphi(u) = (u^a,u^b)$ will be injective for every $k \in \mathbb{Z}_{>0}$.

We may suspect that although the integral's abscissa of convergence may not agree with the algebraic dimension of the variety, the p-adic zeta function's will.

Conjecture 2.10.

$$\forall \mathbf{f} : D_{V(\mathbf{f})} = \sigma_{V(\mathbf{f})} = n-m$$

We shall prove this in a special case.

Theorem 2.11. *Let $f \in \mathbb{Z}_p[x_1,\dots,x_n]$ be such that is homogenenous and $V(f)$ has singularity only in $(0,\dots,0)$, then $D_{V(f)} = \sigma_{V(f)} = n-1$.*

Proof. Let $A_j = p^j\mathbb{Z}_p^n \setminus p^{j+1}\mathbb{Z}_p^n, V_j = V(f)\cap A_j$. Then, since $\mathbb{Z}_p^n = \bigcup_j^* A_j$, we have

$$L(p^k) = 1 + \sum_{j=0}^{k-1} L_{V_j}(p^k),$$

where the 1 comes from $(0,...,0) \mod p^k$.

If $x \in A_j$ then $x = p^j y$, where $y \notin p\mathbb{Z}_p$, meaning that $y \in V_0(f)$. And so, by the homogeneity of f, $f(x) = 0$ if and only if $f(y) = 0$. Hence

$$L_{V_j}(p^k) = N_0(p^{k-j}) = N_0(p)p^{(n-1)(k-j-1)}$$

by Lemma 2.6. Thus

$$L(p^k) = 1 + N_0(p)\sum_{j=0}^{k-1} p^{(n-1)(k-j-1)} = 1 + N_0(p)\frac{p^{(n-1)k}-1}{p^{(n-1)}-1}.$$

This means that

$$\zeta_{V(f)(s)} = \sum_{k=0}^{\infty} \frac{p^{2n-1}-p^n-p^{n-1}+1+N_0(p)((p-1)p^{(n-1)k}-p^n+1)}{(p^{n-1}-1)p^{ks}},$$

hence, the abscissa of convergence is $n-1$.

To obtain the Minkowski dimension it is sufficient to show for $\varepsilon = p^{-l}$. If $x \in \mathbb{Z}_p^n \setminus V(f)$ and $\forall y \in V(f) : |x-y|_p \geq p^{-k}$, then $B_{p^{-k}}(x) \subset \mathbb{Z}_p^n \setminus V(f)$. So to find

$\mathrm{vol}(p^{-l})$, we need to add the measure of the balls with radii at most $p^{-(l+1)}$ in the representation of the fractal string:

$$\mathrm{vol}(p^{-l}) = \sum_{k=l+1}^{\infty} (p^n L(p^{k-1}) - L(p^k))p^{-nk}$$

And so

$$\begin{aligned}\mathrm{vol}(p^{-l}) &= \sum_{k=l+1}^{\infty} \frac{p^{2n-1} - p^n - p^{n-1} + 1 + N_0(p)((p-1)p^{(n-1)k} - p^n + 1)}{(p^{n-1}-1)p^{nk}} \\ &= \frac{Ap^{-n(l-1)}}{p^n - 1} + \frac{Bp^{-(l-1)}}{p-1},\end{aligned}$$

where $A = \frac{p^{2n-1}-p^n-p^{n-1}+1+N_0(p)(1-p^n)}{p^{n-1}-1}$ and $B = \frac{N_0(p)(p-1)}{p^{n-1}-1}$. Thus

$$\lim_{l\to\infty} \log_{p^{-l}}(\mathrm{vol}(p^{-l})) = 1.$$

And so,

$$n - 1 = D_{V(f)} = \sigma_{V(f)}.$$

□

References

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[2] Lapidus, M. L., & Van Frankenhuijsen, M. (2012). Fractal geometry, complex dimensions and zeta functions: geometry and spectra of fractal strings. Springer Science & Business Media.

[3] León-Cardenal, E., & Zúñiga-Galindo, W. A. (2019). An introduction to the theory of local zeta functions from scratch. Revista Integración, 37(1), 45-76.