Igusa local zeta function and fractal strings

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Recap

$\mathbb{R} \quad \longleftrightarrow \quad \mathbb{Z}_p^n$

bounded open set $\longleftrightarrow$ union of countably many disjoint balls

$\mathcal{L} = l_1, l_2, l_3, \ldots \quad \longleftrightarrow \quad V = \bigcup_j^* \bigcup_{i=1}^{*k_j} B_{p^{-j}}(x_{ji}) \implies \{k_j\}_{j=1}^{\infty}$

$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} l_j^s, \quad \longleftrightarrow \quad \zeta_V(s) = \sum_{j=1}^{\infty} k_j p^{-js}$

Theorem (Lapidus, Van Frankenhuijsen)

$$\forall \mathcal{L} : D_{\mathcal{L}} = \sigma_{\mathcal{L}}$$

p-adic fractal zeta function associated to affine varieties

$\mathbf{f} : \mathbb{Z}_p^n \to \mathbb{Z}_p^m$ s.t. $\mathbf{f} = (f_1, \ldots, f_m)$, $f_j \in \mathbb{Z}_p[x_1, \ldots, x_n]$ $(n \geq m)$

Definition

The affine variety of $\mathbf{f}$ is

$$V(\mathbf{f}) = \{x \in \mathbb{Z}_p^n : \mathbf{f}(x) = 0\}.$$

Theorem

When $V(\mathbf{f}) \neq \emptyset$, $\mathbb{Z}_p^n \setminus V(\mathbf{f})$ is a p-adic fractal string.

$$\zeta_{V(\mathbf{f})}(s) = \sum_{k=1}^{\infty} (p^n L(p^{k-1}) - L(p^k)) p^{-ks}$$

$$L(p^k) = |\{\hat{x} \in \mathbb{Z}/p^k\mathbb{Z} | \exists x \in V(\mathbf{f}), [x]_k = \hat{x}\}|$$

$$N(p^k) = |\{x \in (\mathbb{Z}/p^k\mathbb{Z})^n : \mathbf{f}(x) \equiv 0 \mod p^k\}|$$

$$N_0(p^k) = |\{\hat{x} \in \mathbb{Z}/p^k\mathbb{Z} | \exists x \in V(\mathbf{f}) \text{ non-singular over } \mathbb{Z}/p\mathbb{Z}, [x]_k = \hat{x}\}|$$

Proposition

$$\forall k : N_0(p^k) \leq L(p^k) \leq N(p^k)$$

Non-singular varieties

$$\forall k : N_0(p^k) = L(p^k) = N(p^k)$$

Lemma

$$N_0(p^k) = p^{(n-m)(k-1)} N_0(p),$$

- $\sigma_{V(\mathbf{f})} = D_{V(\mathbf{f})} = n - m$
- $\zeta_{V(\mathbf{f})}(s)$ is a rational function of p^{-s}
- $\zeta_{V(f)}(s) = p^{-(n-s)} \int_{\mathbb{Z}_p^n} |f(x)|_p^{-(n-s)} d\lambda(x)$

Example ($f(x, y) = xy$)

$$
\begin{aligned}
N_0(p^2) &= 2p^2 - 2 \\
L(p^2) &= 2p^2 - 1 \\
N(p^2) &= 2p^2
\end{aligned}
$$

Conjecture

$$\forall \mathbf{f} : D_{V(\mathbf{f})} = \sigma_{V(\mathbf{f})} = n - m$$

Example ($f(x,y) = x^2 + y^3$)

$p^{s-2} Z_{V(f)}(s-2) \implies \sigma = 7/6$

$V(f) = \{(t^3, -t^2) : t \in \mathbb{Z}_p\}$

$L(p^k) = |\{(t^3, -t^2) \in \left(\mathbb{Z}/p^k\mathbb{Z}\right)^2\}|$

$p^{k-1}(p-1) \leq L(p^k) \leq p^k$

$$\sigma_{V(f)} = \limsup_{k\to\infty} \frac{\log|\sum_{j=1}^{k} p^2 L(p^{j-1}) - L(p^j)|}{k \log p}$$

Remark

This works for $V(f) = \{(t^a, t^b) : t \in \mathbb{Z}_p\}$*, where* $\gcd(\gcd(a,b), p(p-1)) = 1$*.*

Theorem

Let $f \in \mathbb{Z}_p[x_1, \ldots, x_n]$ be such that is homogenenous and $V(f)$ has singularity only in $(0, \ldots, 0)$, then $D_{V(f)} = \sigma_{V(f)} = n - 1$.

Proof.

$$A_j = p^j \mathbb{Z}_p^n \setminus p^{j+1} \mathbb{Z}_p^n, V_j = V(f) \cap A_j$$

$$L(p^k) = 1 + \sum_{j=0}^{k-1} L_{V_j}(p^k)$$

$$L_{V_j}(p^k) = N_0(p^{k-j}) = N_0(p)p^{(n-1)(k-j-1)}$$

$\square$