(p,q)-Type Theorems in Geometric Settings

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December 2025

1 Introduction

During this semester, I studied some of the main techniques used in discrete geometry. Specifically, my studies focused on piercing problems for families of convex sets satisfying the (p,q)-condition. A family $\mathcal{F}$ of sets in $\mathbb{R}^d$ satisfies the (p,q)-condition if among any p sets in $\mathcal{F}$, there exist q sets with a common point.

For some time, it was an open question whether this property was sufficient to guarantee the existence of finitely many points piercing all sets in a family of convex sets, with $p \geq q \geq d+1$. In 1992, Alon and Kleitman [2] solved this problem.

Theorem 1.1 (The (p,q)-theorem). *Let p, q, d be integers with $p \geq q \geq d+1$. Then there exists a number $HD_d(p,q)$[1] such that if $\mathcal{F} \subseteq \mathbb{R}^d$ is a family of convex sets satisfying the (p,q)-condition, then $\mathcal{F}$ has a transversal consisting of at most $HD_d(p,q)$ points.*

A natural follow-up question is how large $HD_d(p,q)$ must be. Remarkably, the proof of Alon and Kleitman provides bounds that are far from optimal. For instance, in the simplest non-trivial case, a family of convex sets in the plane satisfying the $(4,3)$-condition, their bound is 343. Nevertheless, this bound was lowered to 13 in 2001 by Kleitman, Gyárfás, and Tóth [6]. Recently, McGinnis proved that $HD_2(4,3) \leq 9$, and it is conjectured that $HD_2(4,3) = 3$.

Other variants of the (p,q)-problem consider different properties for the family $\mathcal{F}$, for instance, non-piercing families, families of pseudodisks, or piercing with hyperplanes instead of points.

Theorem 1.2 (A (p,q)-theorem for hyperplane transversals, [1]). *Let $p \geq d+1$, and let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$ such that among every p members of $\mathcal{F}$, there exist $d+1$ that have a common hyperplane transversal (i.e., there is a hyperplane intersecting all of them). Then there exist at most $C = C(p,d)$ hyperplanes whose union intersects all members of $\mathcal{F}$.*

Again, the natural question is what the optimal value of such a constant C is. The technique showcased by McGinnis [9] for the (p,q)-problem also provides a method to prove that if a family of compact connected sets in the plane has the property that every three members are intersected by a line, then there exist three lines intersecting all the sets in the family. In the following section, we briefly discuss such arguments.

[1] This notation refers to Hadwiger and Debrunner, who first raised the problem.

2 The $(4,3)$-problem

As mentioned earlier, the best current bound for the $(4,3)$-problem was given by McGinnis [8]. His approach relies on two main tools.

Theorem 2.1 (KKM theorem [7])**.** *Let $A_1, \dots, A_n$ be open subsets of the simplex Δ^{n-1} such that for every face σ of Δ^{n-1} we have $\sigma \subseteq \bigcup_{e_i \in \sigma} A_i$. Then $\bigcap_{i=1}^n A_i \neq \emptyset$.*

The second tool is a result of Tardos [10] concerning 2-intervals. Let L_1 and L_2 be two homeomorphic copies of $\mathbb{R}$. A *2-interval* is a set of the form $I = I_1 \cup I_2$ where each I_i is an interval in L_i.

Theorem 2.2 (Tardos [10])**.** *If $\mathcal{I}$ is a family of 2-intervals, then $\tau(\mathcal{I}) \leq 2\nu(\mathcal{I})$.*

Given a family $\mathcal{C}$ of convex bodies satisfying the $(4,3)$-property, we rescale it so that every set lies inside the unit circle. For a parametrization $f(t)$ of the circle with $t \in [0,1]$, and a point $x = (x_1, x_2, x_3, x_4) \in \Delta^3$, McGinnis defines a partition of the circle into four regions $R_x^1, \dots, R_x^4$ as in Figure 1.

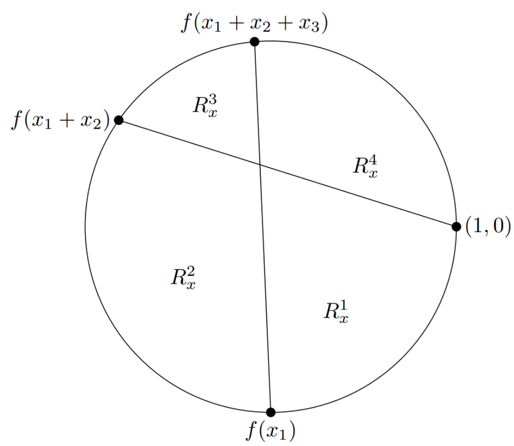

Figure 1: Partition of the unit circle corresponding to a point $x \in \Delta^3$ (from McGinnis [8]).

The regions R_x^j are open and may be empty, but they are always well defined since $x_i \geq 0$ and $\sum_{i=1}^4 x_i = 1$. McGinnis then defines the sets $A_i \subseteq \Delta^3$ by declaring that $x \in A_i$ if there exist three sets $C_1, C_2, C_3 \in \mathcal{C}$ with nonempty intersection and such that all pairwise intersections lie in R_x^i.

Assume that $\Delta^3 \not\subseteq \bigcup_{i=1}^4 A_i$, i.e. there exists $x \in \Delta^3 \setminus \bigcup_{i=1}^4 A_i$. Then the sets in $\mathcal{C}$ that lie entirely inside a region R_x^i can be pierced with two points. Let $\mathcal{D}$ denote the sets in $\mathcal{C}$ that intersect the two dividing segments determined by x. Each $C \in \mathcal{D}$ gives rise to a 2-interval I_C.

Theorem 2.3. *If $x \in \Delta^3 \setminus \bigcup_{i=1}^4 A_i$, then $\tau(\mathcal{D}) \leq 6$.*

Proof. Let $\mathcal{I} = \{I_C : C \in \mathcal{D}\}$. Consider four sets $C_1, C_2, C_3, C_4 \in \mathcal{D}$. By the $(4,3)$-property, some triple, say C_1, C_2, C_3, intersects. Since $x \notin \bigcup_{i=1}^4 A_i$, the intersection $C_1 \cap C_2$ must lie on one of the two dividing segments, implying $I_{C_1} \cap I_{C_2} \neq \emptyset$. Thus $\mathcal{I}$ contains no four pairwise disjoint elements, so $\nu(\mathcal{I}) \leq 3$. Tardos's theorem gives $\tau(\mathcal{I}) \leq 6$. □

Hence, if such an x exists, the family $\mathcal{C}$ can be pierced by at most 8 points. The remaining case is when the sets A_i cover the simplex. Note that these sets satisfy the conditions of the KKM theorem.

Theorem 2.4. *If $\Delta^3 \subseteq \bigcup_{i=1}^4 A_i$, then there exists a point $x \in \bigcap_{i=1}^4 A_i$.*

Fix such an x and consider the regions $R_x^1, \dots, R_x^4$ induced by it. The two dividing segments meet at a point c, which forms the first of the nine piercing points. By convexity, any set not pierced by c must avoid at least one of the regions R_x^i. Let $\mathcal{C}_i$ be the sets avoiding R_x^i. McGinnis then performs a detailed case analysis to show that $\tau(\mathcal{C}_i) \leq 2$ for each $i \in [4]$, yielding a total transversal of size at most 9.

3 Colorful versions

The same technique can be adapted to piercing by lines [9]. In this context, the minimum size of a transversal of lines is called the *line-piercing number*.

Theorem 3.1. *Let $\mathcal{F}_1, \dots, \mathcal{F}_6$ be families of compact connected sets in $\mathbb{R}^2$. If every triple $A_1 \in \mathcal{F}_{i_1}, A_2 \in \mathcal{F}_{i_2}, A_3 \in \mathcal{F}_{i_3}$ with $1 \leq i_1 < i_2 < i_3 \leq 6$ is intersected by a line, then there exists $i \in [6]$ such that the line-piercing number of $\mathcal{F}_i$ is at most 3.*

Remark. *The same statement holds even if only five families are considered [4].*

To prove this, one needs a colorful variant of the KKM theorem due to Gale [5].

Theorem 3.2 (Colorful KKM theorem [5])**.** *For each $i \in [n]$, let $\{A_i^1, \dots, A_i^n\}$ be open sets in the simplex Δ^{n-1} satisfying the KKM condition: for every face σ we have $\sigma \subseteq \bigcup_{e_j \in \sigma} A_i^j$. Then there exists a permutation $\pi \in S_n$ such that $\bigcap_{i=1}^n A_i^{\pi(i)} \neq \emptyset$.*

Moreover, one could wonder whether a colorful version of the (p, q)-theorem holds. This was answered by Bárány, Fodor, Montejano, Oliveros, and Pór [3] in 2014. Let $\mathcal{F}_1, \dots, \mathcal{F}_p$ be finite families of convex sets in $\mathbb{R}^d$, and write $\mathcal{F} = \bigcup_i \mathcal{F}_i$. A heterochromatic p-tuple is a tuple $C_1, \dots, C_p$ with $C_i \in \mathcal{F}_i$. The family $\mathcal{F}$ satisfies the heterochromatic $(p, q)_H$-condition, if every heterochromatic p-tuple of $\mathcal{F}$ contains an intersecting q-tuple.

Theorem 3.3 (Colorful (p, q)-theorem [3])**.** *Let p, q, d be integers with $p \geq q \geq d + 1$. There exists a constant $M(p, q, d)$ such that the following holds: if $\mathcal{F}_1, \dots, \mathcal{F}_p$ satisfy the heterochromatic $(p, q)_H$-condition, then for at least $q - d$ indices $i \in [p]$ we have $\tau(\mathcal{F}_i) \leq M(p, q, d)$.*

4 Future work

In the next semester, I plan to continue studying (p, q)-type problems in different geometric settings and their colorful variants. In particular, I am interested in exploring potential improvements of the current best bound for the $(4, 3)$-problem.

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