(p,q)-Type Theorems in Geometric Settings

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The general problem

A family $\mathcal{F}$ of sets in $\mathbb{R}^d$ satisfies the (p, q)-condition if among any p sets in $\mathcal{F}$, there exist q sets with a common point.

Theorem (The (p, q)-theorem)

Let p, q, d be integers with $p \geq q \geq d + 1$. Then there exists a number $HD_d(p, q)$ such that if $\mathcal{F} \subseteq \mathbb{R}^d$ is a finite family of convex sets satisfying the (p, q)-condition, then $\mathcal{F}$ has a transversal consisting of at most $HD_d(p, q)$ points.

The general problem

Other variants of the (p, q)-problem consider different properties for the family $\mathcal{F}$, for instance, non-piercing families, families of pseudodisks, or piercing with hyperplanes instead of points.

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Theorem (A (p, q)-theorem for hyperplane transversals, [1])

Let $p \geq d + 1$, and let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$ such that among every p members of $\mathcal{F}$, there exist $d + 1$ that have a common hyperplane transversal (i.e., there is a hyperplane intersecting all of them). Then there exist at most $C = C(p, d)$ hyperplanes whose union intersects all members of $\mathcal{F}$.

Fixed Point Iteration Method

The proof of Alon and Kleitman provides bounds that are far from optimal. For instance, in the simplest non-trivial case, a family of convex sets in the plane satisfying the $(4, 3)$-condition, their bound is 343. Nevertheless, this bound was lowered to 13 in 2001 by Kleitman, Gyárfás, and Tóth.

In 2020, McGinnis proved that $HD_2(4, 3) \leq 9$, and it is conjectured that $HD_2(4, 3) = 3$. His method also provides a way to prove that if a family of compact connected sets in the plane has the property that every three members are intersected by a line, then there exist three lines intersecting all the sets in the family.

The KKM Theorem

As mentioned earlier, the best current bound for the $(4, 3)$-problem was given by McGinnis [2]. His approach relies on two main tools.

Theorem (KKM theorem)

Let $A_1, \ldots, A_n$ be open subsets of the simplex Δ^{n-1} such that for every face σ of Δ^{n-1} we have $\sigma \subseteq \bigcup_{e_i \in \sigma} A_i$. Then $\bigcap_{i=1}^{n} A_i \neq \emptyset$.

2-intervals

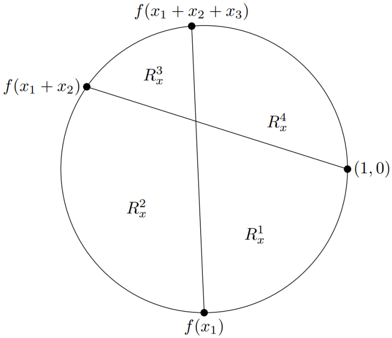
The second tool is a result of Tardos concerning 2-intervals. Let L_1 and L_2 be two homeomorphic copies of $\mathbb{R}$. A *2-interval* is a set of the form $I = I_1 \cup I_2$ where each I_i is an interval in L_i.

Theorem

If $\mathcal{I}$ is a family of 2-intervals, then $\tau(\mathcal{I}) \leq 2\nu(\mathcal{I})$.

Idea of the proof

For a parametrization $f(t)$ of the circle with $t \in [0, 1]$, and a point $x = (x_1, x_2, x_3, x_4) \in \Delta^3$,



Idea of the proof

Let the sets $A_i \subseteq \Delta^3$ be such that $x \in A_i$ if there exist three sets $C_1, C_2, C_3 \in \mathcal{F}$ with nonempty intersection and such that all pairwise intersections lie in R_x^i.

Assume that $\Delta^3 \not\subseteq \bigcup_{i=1}^4 A_i$, i.e. there exists $x \in \Delta^3 \setminus \bigcup_{i=1}^4 A_i$. Then the sets in $\mathcal{F}$ that lie entirely inside a region R_x^i can be pierced with two points. Let $\mathcal{D}$ denote the sets in $\mathcal{F}$ that intersect the two dividing segments determined by x. Each $C \in \mathcal{D}$ gives rise to a 2-interval I_C.

Theorem

If $x \in \Delta^3 \setminus \bigcup_{i=1}^4 A_i$, then $\tau(\mathcal{D}) \leq 6$.

Proof.

Let $\mathcal{I} = \{I_C : C \in \mathcal{D}\}$. Consider four sets $C_1, C_2, C_3, C_4 \in \mathcal{D}$. By the $(4, 3)$-property, some triple, say C_1, C_2, C_3, intersects. Since $x \notin \bigcup_{i=1}^4 A_i$, the intersection $C_1 \cap C_2$ must lie on one of the two dividing segments, implying $I_{C_1} \cap I_{C_2} \neq \emptyset$. Thus $\mathcal{I}$ contains no four pairwise disjoint elements, so $\nu(\mathcal{I}) \leq 3$. Tardos's theorem gives $\tau(\mathcal{I}) \leq 6$. □

Idea of the proof

The remaining case is when the sets A_i cover the simplex. Note that these sets satisfy the conditions of the KKM theorem.

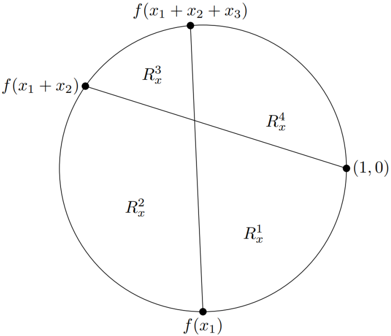
Theorem

If $\Delta^3 \subseteq \bigcup_{i=1}^4 A_i$, *then there exists a point* $x \in \bigcap_{i=1}^4 A_i$.

Fix such an x and consider the regions $R_x^1, \ldots, R_x^4$ induced by it. The two dividing segments meet at a point c, which forms the first of the nine piercing points.

Idea of the proof

By convexity, any set not pierced by c must avoid at least one of the regions R_x^i. Let $\mathcal{C}_i$ be the sets avoiding R_x^i. McGinnis then performs a detailed case analysis to show that $\tau(\mathcal{C}_i) \leq 2$ for each $i \in [4]$, yielding a total transversal of size at most 9.



Colorful Versions

In this context, the minimum size of a transversal of lines is called the *line-piercing number*.

Theorem

Let $\mathcal{F}_1, \ldots, \mathcal{F}_6$ be families of compact connected sets in $\mathbb{R}^2$. If every triple $A_1 \in \mathcal{F}_{i_1}, A_2 \in \mathcal{F}_{i_2}, A_3 \in \mathcal{F}_{i_3}$ with $1 \leq i_1 < i_2 < i_3 \leq 6$ is intersected by a line, then there exists $i \in [6]$ such that the line-piercing number of $\mathcal{F}_i$ is at most 3.

The same statement holds even if only five families are considered [1]. To prove this, one needs a colorful variant of the KKM theorem due to Gale.

Colorful KKM Theorem

Theorem (Colorful KKM theorem, Gale 1984)

For each $i \in [n]$, let $\{A_i^1, \ldots, A_i^n\}$ be open sets in the simplex Δ^{n-1} satisfying the KKM condition: for every face σ we have $\sigma \subseteq \bigcup_{e_j \in \sigma} A_i^j$. Then there exists a permutation $\pi \in S_n$ such that $\bigcap_{i=1}^n A_i^{\pi(i)} \neq \emptyset$.

Moreover, one may wonder whether a colorful version of the (p, q)-theorem holds. This was answered by Bárány, Fodor, Montejano, Oliveros, and Pór [3] in 2014.

Let $\mathcal{F}_1, \ldots, \mathcal{F}_p$ be finite families of convex sets in $\mathbb{R}^d$, and write $\mathcal{F} = \bigcup_i \mathcal{F}_i$. A heterochromatic p-tuple is a tuple $C_1, \ldots, C_p$ with $C_i \in \mathcal{F}_i$. The family $\mathcal{F}$ satisfies the heterochromatic $(p, q)_H$-condition, if every heterochromatic p-tuple of $\mathcal{F}$ contains an intersecting q-tuple.

Colorful (p, q)-theorem

Theorem (Colorful (p, q)-theorem, [3])

Let p, q, d be integers with $p \geq q \geq d + 1$. There exists a constant $M(p, q, d)$ such that the following holds: if $\mathcal{F}_1, \ldots, \mathcal{F}_p$ satisfy the heterochromatic $(p, q)_H$-condition, then for at least $q - d$ indices $i \in [p]$ we have $\tau(\mathcal{F}_i) \leq M(p, q, d)$.

Future work

In the next semester, I plan to continue studying (p, q)-type problems in different geometric settings and their colorful variants. In particular, I am interested in exploring potential improvements of the current best bound for the $(4, 3)$-problem.

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