Monochromatic Monotone Path Problems and (3,2)-Sequences

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Some famous problems

Theorem (Erdős–Szekeres)

For any positive integers A, B, every sequence of length $AB + 1$ contains either an increasing subsequence of length $A + 1$ or a decreasing subsequence of length $B + 1$.

Happy Ending Problem (Erdős–Szekeres)

For every $n \geq 3$, let $N(n)$ be the smallest number such that any set of $N(n)$ points in the plane in general position contains n points in convex position. ($N(n)$ is known to be finite.)

Conjecture:

$$N(n) = 2^{n-2} + 1.$$

- Motivation: analyse these in a more generic framework

Monochromatic Monotone Path Problem

Definition

Let $N_k(q, n)$ be the smallest integer N such that every q-coloring of the edges of the complete k-uniform hypergraph on the ordered vertex set $\{1, \ldots, N\}$ contains a monochromatic monotone path of length n.

- A monotone path is just a sequence $i_1 < i_2 < \cdots < i_m$ such that all edges $(i_j, i_{j+1}, ..., i_{j+k-1})$ have the same color
- The classical EST corresponds to bounding $N_2(2, n)$
  - EST is also transitive, but this does not change the answer here
- For $k > 3$ it is unknown whether transitivity affects $N_k(q, n)$
- General bounding techniques:
  - vertex labeling via down-sets
  - pigeonhole principle

The General Monotone Sequence Problem

Definition

A sequence $x_1, \ldots, x_L$ with $x_i \in [n]^r$ is *s-increasing* if for any $i < j$, at least s coordinates satisfy $x_{i,k} < x_{j,k}$. Let $L(r, s; n)$ denote the maximum possible length L of such an s-increasing sequence.

- Directly related to the hypergraph problem
  - e.g. $s = 1$, $r = q$ gives a q-coloring interpretation for the 2-uniform (graph) case

The $(3,2)$-Sequence Problem

- We focus on $F(n) = L(3,2;n)$ (2 out of 3 coordinates strictly increase between any two elements of the sequence)
- Known asymptotic bounds:

$$\Omega(n^{3/2}) \leq F(n) \leq O(n^{2-\varepsilon}) \quad \text{for some } \varepsilon > 0$$

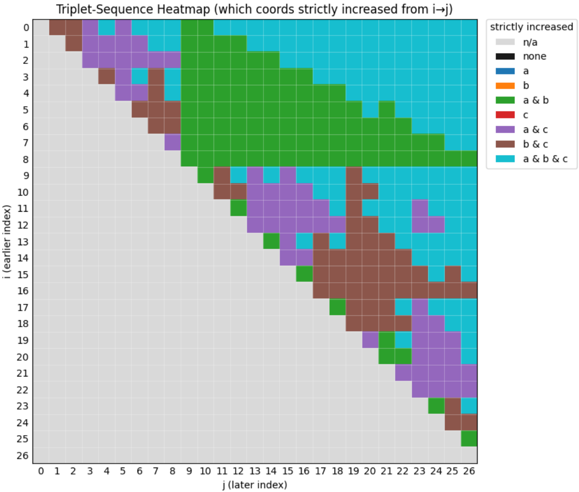
- A nice construction for perfect squares gives the lower bound
- $O(n^2)$ upper bound trivially follows from EST

Computing F(n) values

- Computed exact values of $F(n)$ for $n \leq 9$
  - Motivation: any construction with length greater than $n^{3/2}$ can be generalised for larger n values and results in a lower bound improvement
- Verified that $F(9) = 27$ and computed all 874,776 optimal constructions with a C++ program
  - Motivation: help to generate and verify conjectures

Optimal Construction Example

- Example heatmap of a randomly chosen optimal $(3, 2)$-sequence for $n = 9$



Conjectures and Conditional Results

- **Conditional result:** if one coordinate of a $(3,2)$-sequence is monotone increasing, then

$$F(n) \leq n^{3/2}.$$

- **Uniformity Conjecture:** if n is a perfect square, every optimal $(3,2)$-sequence uses each value in $[n]$ exactly $\sqrt{n}$ times in each coordinate

Definition (Cut)

A sequence of triples $x_1, x_2, \ldots, x_L$ is said to have a *cut* in the kth coordinate if there exists an index $i \in \{1, \ldots, L-1\}$ such that for all $j \leq i$ and $m > i$,

$$x_{j,k} < x_{m,k}.$$

- **Cut Conjecture:** every optimal $(3,2)$-sequence has a cut in at least one coordinate

Future Work

- Try to improve one of the bounds for $F(n)$
- Extend methods to higher dimensions

Thank You!

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Disclosure

AI Disclosure: I used ChatGPT to assist with LaTeX templating and to refine the clarity and phrasing of the text.