Arborescence packing

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December 14, 2025

My goal in this short note is to present some well-known and useful theorem from the topic of arborescence-packing, more specially the ones which are related to Edmonds' theorems. I also show an application of arborescences in routing networks.

Arborescences are directed trees, where every vertex has indegree 1, but the root. Equivalently, we can define an arborescence to be a rooted directed tree in which every vertex can be reach from the root on (a unique) directed path. If the root is $s \in V$, then it is called an s-arborescence.

There are two main theorems from Edmonds and Tutte. Edmonds' (weak) theorem says that in a directed graph there are k pairwise arc-disjoint spanning s-arborescences iff for every vertex there are k arc-disjoint paths from s to the vertex (the graph is rooted k-edge-connected). Tutte's theorem is for an undirected graph and spanning trees, and the existence depends on the number of edges between the members of a partition of the vertex set.

Let R be a nonempty subset of V, then the R-branching means a branching T (disjoint union of arborescences) where the rootset is R. If $R_1, R_2, ..., R_k$ are nonempty subsets of V, then the $R_1, R_2, ..., R_k$-branching means that every branching T_i is a R_i-branching. A branching T_i is spanning, if every vertex of V is covered by the branching T_i. *Edmonds' branching theorem* gives a necessary and sufficient condition for the existence of arc-disjoint spanning $R_1, R_2, ..., R_k$-branchings in a directed graph.

Even if we cannot reach every vertex from the rootsets, we can take the reachability R_i-branching, which means that each R_i-branching only spans the vertices which are reachable from the vertices of R_i. The theorem of *Katoh, Kamiyama, Takizawa* characterize this problem, and it is very similar to the previous one. It follows from *Edmonds' branching theorem.* There are similar theorems about forests in undirected graphs too.

These are the main theorems in the topic. Now we present some results of routing networks using arborescences, based on [1], [2]. The following results are maybe not so useful in general, but they give some interesting examples and tricks how to use arborescences as a routing function.

The model is the following: given an undirected multigraph $G = (V, E)$, where each vertex is a router of the network and there is a unique vertex, called destination d. The (undirected) edges are the links of the network and we denote an undirected edge between x and y by $\{x, y\}$. Because we want to present a network, we will create a directed copy of G, call it $\overleftrightarrow{G}$. Every edge will be added in two direction such that $\{x, y\} \in E(G)$ if and only if $(x, y), (y, x) \in E(\overleftrightarrow{G})$, where (x, y) denote the arc from x to y. We say that arborescences T and T' are arc-disjoint, if $\vec{e} \in T$ ($\vec{e}$ is an (x, y) arc) implies that $\vec{e} \notin E(T')$ (and $\vec{e} \in E(T') \Rightarrow \vec{e} \notin E(T)$). We say that T and T' do not share an edge $\{x, y\} \in E$, if $\vec{e} \in E(T)$ implies that $\vec{e}$ and $\overleftarrow{e}$ are not arcs of T'. If they do not share any edges, then we call them

edge-disjoint.
Here we will use in-arborescences, while in the previous theorems they were out-arborescences, but it is easy to see, that by reversing the orientation of the edges we have similar results. We have to send a packet from (every) vertex to d. If the graph is connected, than it is easy. However there can be failed edges, when the packet cannot be sent through an edge (in both direction). We do not know which edges will be failed, so we should create a routing function which works in every cases. In a vertex there are rules like, if we used the arc (u, v), then we will continue the packet forwarding on the arc (v, w), but this has to be settled without knowing the failed edges. If f edges are failed and we still can forward the packet from any vertex to d, then we say that the routing function is f-resilient.
The main routing function during the short note is the *circular-in-arborescence routing.*

Definition 1. *([1], [2]) The circular-in-arborescence routing means we have a $T = \{T_1, T_2, ...T_k\}$ set of k d-rooted arc-disjoint spanning in-arborescences of $\overleftrightarrow{G}$. We follow the sequence of the in-arborescences. First the packet is routed through the unique directed path of T_1 towards the destination. If the packet hits a failed arc (edge) at vertex v along T_1, then we reroute along T_2 towards the destination. If we rout along T_i and the packet hits a failed arc at vertex u, then we reroute along $T_{(i+1) \bmod k}$.*

It seems to be an easy and sufficient routing function, however, it is not true, that circular-in-arborescence routing is sufficient to achieve 3-resiliency when the graph is not 4-edge-connected ([1], *Figure* 1.). However, if the graph G is k-edge-connected, where $k = 2, 3, 4, 5$ then the circular-in-arborescence routing is useful, and it is $k-1$-resilient routing. In the first two cases the proofs are very simple, they come from Edmonds' theorem. If we have a 4-edge-connected graph, then we need a stronger condition. This is because, in this case there is a 4-edge-connected graph with 4 arc-disjoint d-rooted spanning in-arborescences, where any circular-in-arborescence routing of them fails ([1] *A*.2) So we need a stronger condition.
Mader's theorem says that k-edge-connected directed graphs can be built up by adding new edges and pinching k existing edges. This theorem was extended to unoriented graphs by László Lovász.

Proposition 1. *([1]) G is an undirected, k-edge-connected graph, it has a vertex d. Suppose that in $\overleftrightarrow{G}$ there exist k arc-disjoint spanning in-arborescences $T_1, ..., T_k$ rooted at d such that, if k is even (odd), $T_1, ..., T_{\frac{k}{2}}$ ($T_1, ..., T_{\lfloor \frac{k}{2} \rfloor}$) are edge-disjoint with each other and $T_{\frac{k}{2}+1}, ..., T_k$ ($T_{\lfloor \frac{k}{2} \rfloor+1}, ..., T_k$) are edge-disjoint with each other. If this holds, then if G' graph is obtained from G by applying the edge-pinching and edge-adding operations then the oriented version of G' ($\overleftrightarrow{G'}$) has k d-rooted in-arborescences with the previous conditions.*

Remark: G is k-edge-connected graph, d is a given vertex of G. We start from d and another vertex v and we connect them with k parallel edges, so this basic graph G_0 is k-edge-connected. Orient these edges from v to d, then they will be in $\overleftrightarrow{G_0}$ k arc-disjoint d-rooted in-arborescences, which are edge-disjoint too. Then according to Proposition 1 during the operations which build up G, in the directed version of the graph there are always k special in-arborescences (defined in the previous proposition), so $\overleftrightarrow{G}$ will also have them.
From this we get a very similar result to *Edmonds' weak theorem* for $\overleftrightarrow{G}$, but it is stronger, if G is k-edge-connected.

Proposition 2. *([1]) For any undirected k-edge-connected graph G, with $k \geq 1$, and any vertex $d \in V$, in $\overleftrightarrow{G}$ there exist k arc-disjoint spanning in-arborescences $T_1, ..., T_k$ rooted at d such that, if k is even (odd), $T_1, ..., T_{\frac{k}{2}}$ ($T_1, ..., T_{\lfloor \frac{k}{2} \rfloor}$) are edge-disjoint with each other and $T_{\frac{k}{2}+1}, ..., T_k$ ($T_{\lfloor \frac{k}{2} \rfloor+1}, ..., T_k$) are edge-disjoint with each other.*

This is the needed stonger condition for 4-edge-connected graphs. In the circular-in-arborescence routing we will use these in-arborescences, but not in this order. Always the second ones will be edge-disjoint. Then the circular-in-arborescence routing is 3-resilient for any undirected 4-edge-connected graph.
Unfortunately, if $k \geq 5$ then a circular-in-arborescence-routing does not always exist, even with these special in-arborescences. Here is another theorem which can be useful with higher k, however it will not solve the problem in general.

Theorem 1. *([1], [2]) Assume that G is an undirected k-edge-connected graph and $\overleftrightarrow{G}$ has k arc-disjoint in-arborescences: $T_1, ..., T_{k-1}, T_k$. Futhermore, the circular-in-arborescence routing based on $T_1, ...T_{k-1}$ is $c-1$-resilient, where $c < k$. Then there exists routing which is c-resilient.*

Routing: We start the routing along T_k. If the packet hits a failed edge $\{x, y\}$ and the arc was $\vec{e} = (x, y)$ then we will forward the packet along T_i which contains $\overleftarrow{e} = (y, x)$. If none of $T_1, ..., T_{k-1}$ contains $\overleftarrow{e}$, then the packet will be forwarded along T_1. This way the packet can be sent from vertex x. After this we forget the in-arborescence T_k and then if the packet hits a failed edge, it will be routed along the next in-arborescence in the circular-order, until it reaches d.
$\vec{e} \in E(T_k)$, so none of the other arborescences will contain it. After hitting the failed $\{x, y\}$ edge, we will route along an in-arborescence which contains the edge in the other direction (and there is at most one T_i) or none of them will use this edge, so routing along $T_1, ...T_{k-1}$ arborescences, the packet will not hit this failed edge again. There are $c-1$ other failed edges, but the circular-in-arborescence routing based on $T_1, ...T_{k-1}$ is $c-1$-resilient, so this routing will be c-resilient.

Proposition 3. *If $\overleftrightarrow{G}$ has k arc-disjoint in-arborescences and for $T_1, ...T_{k-1}$ the circular-in-arborescence routing is $k-2$ resilient, then with the in-arborescence T_k, we get a $k-1$-resilient routing, using the previous technique.*

From this comes the result, there is a 4-resilient routing for 5-edge-connected graphs. Although, we have seen two interesting routing function, which use circular-in-arborescence routing, we cannot use them in the case of $k \geq 6$, since in case $k = 5$ we only have a routing function, but it is not circular. However, in special cases, when we know there are k arc-disjoint in-arborescences and the circular-in-arborescence routing based on $k-1$ of them, is $c-1$-resilient we can always get c-resilient routing function.
There are two final results. The first one comes from *Edmonds' weak theorem.*

Proposition 4. *([2]) For any undirected, k-edge-connected G graph, there exist a set of $\lceil \frac{k}{2} \rceil - 1$-resilient circular-in-arborescence routing.*

From this and Theorem 1, the following is true.

Proposition 5. *([2]) For any undirected, k-edge-connected G graph, there exist a set of $\lfloor \frac{k}{2} \rfloor$-resilient forwarding function.*

There are several extensions of *Edmonds' theorem* (e.g. with matroid roots, directed hypergraphs, mixed graphs). My goal is to find other applications of *Edmonds' theorem* and check whether the new, stronger versions (like the above mentioned theorems) can be used to extend these applications.

References

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[2] Marco Chiesa, Ilya Nikolaevskiy, Slobodan Mitrović, Andrei Gurtov, Aleksander Madry, Michael Schapira, and Scott Shenker. On the resiliency of static forwarding tables. *IEEE/ACM Transactions on Networking*, 25(2):1133–1146, 2016.