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The activity of the stochastic chip-firing game

Individual Project 1
Report

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1 Introduction

As a continuation of my thesis, the topic of my Student Project in this semester was the chip-firing game which is a one-player game on a loopless undirected graph (it can be studied on directed graphs too but it is not a topic of this report). Usually this graph is connected and in this report we consider them to be without parallel edges too. At the beginning of the game, a nonnegative integer is assigned to every vertex which is meant to represent the number of chips the vertex has. A vertex is active if it has at least as many chips as its degree. An active vertex can be fired, which means moving one chip from this vertex along each edge incident to it. As a result, each neighboring vertex gets one chip. The first version of this game (let this game be called asynchronous) - in which exactly one active node fires at the same time - was invented in the 1980s and other versions have been studied since. One of these is parallel chip-firing, in which every active vertex fires simultaneously. In the stochastic chip-firing game, the firing vertex is chosen uniformly at random among the active nodes.

2 Activity

The asynchronous chip firing game is not deterministic, but certain characteristics are independent of the order in which the vertices fire. One of these properties is the finiteness of the game or the number of times a vertex fires until the end of the game (if it stops). The average number of active nodes throughout the game, the so-called activity, however, depends heavily on the firings in the case of non-terminating games. In my thesis, I studied the activity for trees in the different versions of the game. For the stochastic chip-firing on trees with m chips (m is the number of edges), I provided and proved a formula for the star graph and conjectured a formula for the path graph (this formula is $\frac{m^2+3m-2}{4m-2}$) and defined a degree-sequence poset, on which the activity was conjectured to increase. This would mean that among the trees on n nodes, S_n has the highest, while P_n has the lowest activity in the stochastic chip-firing game. In this semester, I studied the stochastic activity of the path graph, and the activity of trees in the asynchronous chip-firing game.

2.1 Activity of the path graph in the stochastic chip-firing game

As the stochastic chip-firing game can be viewed as an aperiodic, irreducible Markov-chain on the recurrent states of chip configurations, it has a unique stationary distribution. A chip configuration is recurrent if there exists a nonempty sequence of firings in the asynchronous game that transforms it back to itself. It has been proven before that for an undirected graph G, a chip configuration x is recurrent if and only if there exist an orientation of the edges of G, for which every vertex has at least as many chips in x as its in-degree in the orientation. So for a path with m chips, the number of recurrent states is 2^m. The stochastic activity therefore equals $\sum_{i=1}^{2^n} \pi_i \cdot a(i)$, where for each recurrent chip configuration i, π_i denotes its stationary probability and $a(i)$ denotes the number of active nodes in the chip configuration. For a path, we can assign an m-long binary string to each state, where the i-th digit is 0 if the i-th edge of the path is oriented from left to right, and it is 1 otherwise. So if we order these states by their binary equivalent lexicographically, the first state is when all edges are oriented from left to right. An inner node is active if the corresponding two digit-string is 01 in the binary string. The first node is active if the first edge is oriented from right to left (so the binary string starts with 1), while the last is active if the last edge is oriented from left to right (the binary

string ends in 0).
Since the activity formula for the path equals the dot product of the 2^m dimensional activity vector and the 2^m dimensional stationary distribution vector, it is worth to take a look at these vectors. It is easy to notice that one can easily obtain the 2^{m+1}-dimensional activity vector from the 2^m-dimensional one: its first 2^m entries are the same as the 2^m-dimensional vector, the next 2^{m-1} entries are the same as the first half of the 2^m entries, but $+1$ coordinate-wise, and the last 2^{m-1} entries are just copied again from the second half of the 2^m-dimensional vector. Similar recursions can be obtained for the transition matrices of the Markov-chains on the recurrent chip configurations. Also, I have made the following conjectures by analyzing the stationary distributions before normalization for different m-s. Every coordinate is a polynomial in m, where m is the number of edges the path has [Table 1]. However, a general two-variable formula for the k-th stationary coordinate for the path with m edges is yet to be found.

Coordinate in the Stationary Distribution	**Polynomial in m before normalization**
1	1
2	$\binom{m}{1}$
3	$2(m-1)$
4	$\binom{m}{2}$
5	$2(m-2)$
6	$(m-2)(m+1)$
7	$(m-2)(m-1)$
8	$\binom{m}{3}$
9	$2(m-3)$

Table 1: Polynomials of the Coordinates Expressed in m

Another way to calculate the activity is to add the activities of the vertices, which can be defined as the probability that they are active in the long run, so it is the sum of the stationary probabilities of the states in which they are active. It is easy to find a pattern for the activities of the vertex. The conjecture is that for the 2 nodes at the ends of the path, their activity is $\frac{C_m+(m-1)C_{m-1}}{\binom{2m}{m}}$, while the activity of the inner nodes is $\frac{\binom{2m-2}{m}\cdot\frac{m+4}{m+1}}{\binom{2m}{n}}$, where C_m denote the m-th Catalan-number. This would of course imply the interesting result, that all inner vertices have the same activity, which might be a little counterintuitive since one could think that the activity depends on their position.
Regarding the equal inner activities, one thing that I proved is that in the stationary distribution, all vertices have the same probability to fire, so each vertex fires with probability $\frac{1}{n}$. This can be seen the following way: let f_r denote the number of 1-s in the first r digits of a state. It is easy to see that if the first vertex fires, then f_r decreases by 1, if the $(r+1)$-th vertex fires, then f_r increases by 1, otherwise the value of f_r does not change, since any inside firing makes a local 01 to 10, and any outside firings does not affect the number of 1-s among the first r edges. Since for the stationary distribution, the expected value of Δf_r is 0, so $0 = \mathbf{E}(\Delta f_r) = P((r+1)\text{-th vertex fires}) - P(\text{first vertex fires})$ which means that $P((r+1)\text{-th vertex fires}) = P(\text{first vertex fires})$ for every r, so we know that every node has the same probability of firing. It is easy to observe that a similar argument can be used for any tree.