# The activity of the stochastic chip-firing game

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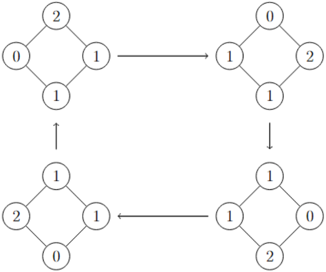
# Summary

1. Introduction
2. Activity of the stochastic chip-firing game
3. Stochastic activity of the path graph
4. Further plans

# Introduction

- Dynamics of the game
- Parallel chip-firing
- Asynchronous chip-firing
- Stochastic chip-firing



# Activity of the stochastic chip-firing game

## Definition

A vertex is **active**, if it has at least as many chips as its degree.

## Activity of the stochastic chip-firing game

For a $G$ graph and $x$ initial chip-distribution, the activity of the stochastic chip-firing game is the expected value of the average number of active vertices, that is, $A_G(x) = \lim_{k\to\infty} \frac{1}{k} E(A_k(x))$, where $A_k(x)$ is the random variable denoting the total number of active vertices in the first $k$ steps of the game started from the chip configuration $x$.

# Stochastic chip-firing game on trees with $n$ vertices and $n-1$ chips

## Proposition

The activity of the stochastic chip-firing game on a given tree does not depend on the initial chip configuration, only on the number of chips.

## Star

The activity of the stochastic chip-firing game with $n-1$ chips on a star with $n$ vertices is always

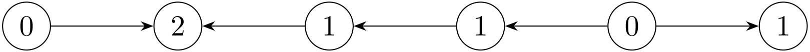
$$A_{S_n} = \frac{n^2 - n + 2}{2n}.$$

## Conjecture for path graphs

The activity of the stochastic chip-firing game with $n-1$ chips on a path with $n$ vertices is always $A_{P_n} = \frac{n^2+n-4}{4n-6}$.

# Stochastic activity of the path graph

- Recurrent chip configurations: arbitrary orientation of the edges of the tree, for each vertex: number of chips = indegree $\implies 2^m$ recurrent states for the path graph with $m$ chips




- Aperiodic, irreducible Markov-chain on the recurrent states of chip configurations $\implies$ stationary vector $\pi$

- Activity $= \pi \cdot a$, where $a$ denotes the number of active vertices for the $2^m$ states

# Stationary Distribution

| **Coordinate in the Stationary Distribution** | **Polynomial in $m$ before normalization** |
|---|---|
| 1 | $1$ |
| 2 | $\binom{m}{1}$ |
| 3 | $2(m-1)$ |
| 4 | $\binom{m}{2}$ |
| 5 | $2(m-2)$ |
| 6 | $(m-2)(m+1)$ |
| 7 | $(m-2)(m-1)$ |
| 8 | $\binom{m}{3}$ |
| 9 | $2(m-3)$ |

Table 1: Polynomials of the Coordinates Expressed in $m$

# Another way to calculate the activity

- Activity = sum of the activities of the vertices
- Conjectures for the activity of the nodes
  - Inner vertices: $\frac{C_m+(m-1)C_{m-1}}{\binom{2m}{m}}$, where $C_m$ denotes the $m$-th Catalan-number
  - The two vertices at the end of the path graph: $\frac{\binom{2m-2}{m}\cdot\frac{m+4}{m+1}}{\binom{2m}{n}}$
- In stationary distribution, every vertex has the same probability of firing (so $\frac{1}{n}$)

# Further plans

- Finishing the proof for the path graph
- Another (much more complex) conjecture left to prove: the activity is increasing on a poset of trees on $n$ vertices

Thank you for your kind attention!