Motivation

The Korteweg–de Vries (KdV) equation is a classical model for the unidirectional propagation of long, small-amplitude waves in shallow water.

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1}$$

Equation (1) possesses soliton solutions that propagate without changing shape and interact in an almost elastic way. From the viewpoint of numerical analysis, such qualitative features are delicate: naive discretizations tend to destroy them in long-time simulations. For reliable long-time computations it is therefore not sufficient to approximate the differential operators alone; one should also take into account the underlying geometric structure.

A particularly important structure is the Hamiltonian formulation of KdV. In this setting, the equation is viewed as an infinite-dimensional analogue of a finite-dimensional Hamiltonian system, endowed with a Hamiltonian functional and a Poisson operator. In particular, it motivates the construction of space and time discretizations that preserve, in a suitable sense, the Hamiltonian or the associated symplectic/Poisson structure.

From finite-dimensional Hamiltonian systems to Hamiltonian PDEs

Finite-dimensional case: Hamiltonian ODEs

In the finite-dimensional setting, the phase space is $\mathbb{R}^{2d}$ and the state is a vector $y = (p, q) \in \mathbb{R}^{2d}$. A Hamiltonian system of ordinary differential equations in canonical form is given by

$$\dot{y} = J^{-1}\nabla H(y), \qquad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{2}$$

where $H : \mathbb{R}^{2d} \to \mathbb{R}$ is the Hamiltonian (typically the total energy) and J is the standard symplectic matrix.

Two fundamental properties follow immediately from (2):

- *Energy conservation.* If $y(t)$ is a solution of (2), then

$$\frac{d}{dt}H(y(t)) = \nabla H(y(t))^{\mathsf{T}}\dot{y}(t) = \nabla H(y(t))^{\mathsf{T}}J^{-1}\nabla H(y(t)) = 0.$$

- *Symplecticity of the flow.* The flow map $\Phi^t : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ of (2) satisfies

$$\big(\Phi'(y)\big)^{\mathsf{T}} J\, \Phi'(y) = J \quad \text{for all } y.$$

  A differentiable map $\Phi : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ with this property is called a *symplectic map.*

Infinite-dimensional case: Hamiltonian PDEs

Hamiltonian partial differential equations can be viewed as infinite-dimensional analogues of Hamiltonian ODEs. The main differences are:

- The phase space is no longer the finite-dimensional space $\mathbb{R}^{2d}$, but a function space X, typically a Hilbert space of functions on a spatial domain Ω (for example $X = L^2(\mathbb{R})$ or a suitable Sobolev space).
- The structure matrix J is replaced by a skew-adjoint linear operator $J : X \to X$ (for instance $J = -\partial_x$ with periodic or decaying boundary conditions).
- The gradient ∇H is replaced by the variational derivative $\delta H/\delta u$, which is the infinite-dimensional analogue of the gradient with respect to the L^2 inner product.

In the infinite-dimensional setting, the Hamiltonian is a functional

$$H[u] = \int_\Omega \mathcal{H}(u, u_x, u_{xx}, \dots)\, dx,$$

where $\mathcal{H}$ is a Hamiltonian density depending on u and finitely many spatial derivatives.

The structural operator is a skew-adjoint linear operator

$$J : X \to X, \qquad \langle Ju, v\rangle = -\langle u, Jv\rangle \quad \text{for all } u, v \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the L^2 inner product. A typical example in one space dimension is $J = -\partial_x$.

The *variational derivative* $\delta H/\delta u$ is defined implicitly by

$$\left\langle \frac{\delta H}{\delta u}, v \right\rangle = \frac{d}{d\varepsilon} H[u + \varepsilon v]\Big|_{\varepsilon=0} \qquad \text{for all } v \in X. \tag{3}$$

As a simple example, consider

$$H[u] = \int_\Omega \frac{1}{2} u^2 \, dx.$$

Then

$$\frac{d}{d\varepsilon} H[u + \varepsilon v]\Big|_{\varepsilon=0} = \int_\Omega u \, v \, dx,$$

so (3) implies

$$\frac{\delta H}{\delta u} = u.$$

This shows that the variational derivative is the L^2-gradient of the functional.

With these notions, a *Hamiltonian PDE* is an evolution equation of the form

$$u_t = J \, \frac{\delta H}{\delta u}. \tag{4}$$

If $u(t)$ is a sufficiently regular solution of (4), then

$$\frac{d}{dt} H[u(t)] = \Big\langle \frac{\delta H}{\delta u}, u_t \Big\rangle = \Big\langle \frac{\delta H}{\delta u}, J \frac{\delta H}{\delta u} \Big\rangle = -\Big\langle J \frac{\delta H}{\delta u}, \frac{\delta H}{\delta u} \Big\rangle = -\frac{d}{dt} H[u(t)],$$

so the derivative must vanish and $H[u(t)]$ is conserved in time.

The KdV equation in Hamiltonian form

We now apply the abstract framework above to the KdV equation (1). We consider

$$u_t + 6uu_x + u_{xxx} = 0, \qquad x \in \mathbb{R}, \; t > 0,$$

and assume that $u(x,t)$ and its derivatives decay sufficiently fast as $|x| \to \infty$.

We wish to show that KdV can be written in the Hamiltonian form (4). To this end we define the Hamiltonian functional

$$H[u] = \int_{\mathbb{R}} \left(\frac{u^3}{3} - 3u_x^2 \right) dx,$$

and choose the skew-adjoint operator

$$J = D = -\partial_x.$$

Step 1: computation of the variational derivative. For an arbitrary direction v we have

$$\Big\langle \frac{\delta H}{\delta u}, v \Big\rangle = \frac{d}{d\varepsilon} H[u + \varepsilon v]\Big|_{\varepsilon=0}.$$

Inserting the explicit expression for H and expanding in ε yields

$$\begin{aligned} \Big\langle \frac{\delta H}{\delta u}, v \Big\rangle &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\mathbb{R}} \left[\frac{(u + \varepsilon v)^3 - u^3}{3} - 3\big((u_x + \varepsilon v_x)^2 - u_x^2\big) \right] dx \\ &= \int_{\mathbb{R}} \big(u^2 v - 6u_x v_x\big) \, dx. \end{aligned}$$

We integrate the second term by parts. Under our decay assumptions the boundary terms vanish, and we obtain

$$\int_{\mathbb{R}} -6u_x v_x \, dx = \int_{\mathbb{R}} 6u_{xx} v \, dx.$$

Thus

$$\Big\langle \frac{\delta H}{\delta u}, v \Big\rangle = \int_{\mathbb{R}} (3u^2 + u_{xx}) \, v \, dx \quad \text{for all } v,$$

which implies

$$\frac{\delta H}{\delta u} = 3u^2 + u_{xx}. \tag{5}$$

Step 2: recovering KdV from the Hamiltonian structure. The Hamiltonian PDE associated with H and $J = -\partial_x$ is

$$u_t = J \, \frac{\delta H}{\delta u} = -\partial_x (3u^2 + u_{xx}).$$

Using (5) we obtain

$$u_t = -6uu_x - u_{xxx},$$

which is equivalent to the KdV equation (1). We have therefore written KdV in Hamiltonian form

$$u_t = J \, \frac{\delta H}{\delta u}, \qquad J = -\partial_x, \quad H[u] = \int_{\mathbb{R}} \left(\frac{u^3}{3} - 3u_x^2 \right) dx. \tag{6}$$