The Hamiltonian Structure of the Korteweg–de Vries Equation

Sidan Jia

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Outline

Motivation

The Korteweg–de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0$$

is a classical model for the propagation of long, small-amplitude waves in shallow water.

It possesses soliton solutions that propagate without changing shape. From the numerical point of view, such qualitative properties are delicate and are often destroyed by naive discretization methods.

This motivates the study of the *Hamiltonian structure* of KdV, which provides a solid basis for the construction of efficient and stable numerical schemes.

Hamiltonian Partial Differential Equations

Hamiltonian partial differential equations are infinite-dimensional analogues of Hamiltonian systems.

The phase space is a Hilbert space

$$X \subset L^2(\mathbb{R}),$$

and the Hamiltonian is a functional

$$H : X \longrightarrow \mathbb{R}.$$

The evolution equation is given by

$$u_t = J \frac{\delta H}{\delta u},$$

where $J : X \to X$ is a skew-adjoint linear operator and $\delta H / \delta u$ denotes the variational derivative.

Skew-Adjoint Operators

A linear operator $J : X \to X$ is called *skew-adjoint* if

$$\langle Ju, v\rangle = -\langle u, Jv\rangle \quad \text{for all } u, v \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the L^2 inner product.

A fundamental example is the spatial derivative operator

$$J = -\partial_x.$$

Indeed, for sufficiently decaying functions u and v,

$$\langle -u_x, v\rangle = \int_{\mathbb{R}} -u_x v \, dx = \int_{\mathbb{R}} u v_x \, dx = -\langle u, -v_x\rangle,$$

so $-\partial_x$ is skew-adjoint.

Hamiltonian Formulation of KdV

Our goal is to find an appropriate Hamiltonian functional

$$H : X \longrightarrow \mathbb{R}$$

such that the associated Hamiltonian PDE

$$u_t = J \frac{\delta H}{\delta u}$$

reproduces the KdV equation.

We define

$$H[u] = \int_{\mathbb{R}} \left(\frac{u^3}{3} - 3u_x^2 \right) dx, \qquad J = -\partial_x.$$

Variational Derivative

To compute the variational derivative, we consider

$$\left\langle \frac{\delta H}{\delta u}, v \right\rangle = \frac{d}{d\varepsilon} H[u + \varepsilon v] \Big|_{\varepsilon=0}.$$

A direct computation yields

$$\frac{\delta H}{\delta u} = 3u^2 + u_{xx}.$$

Recovering the KdV Equation

Substituting the variational derivative into the Hamiltonian PDE, we obtain

$$u_t = -\partial_x(3u^2 + u_{xx}) = -6uu_x - u_{xxx}.$$

This is exactly the Korteweg–de Vries equation.

Therefore, KdV can be written in Hamiltonian form as

$$u_t = J\frac{\delta H}{\delta u}, \qquad J = -\partial_x.$$

Conclusion

- The KdV equation admits a Hamiltonian formulation in an infinite-dimensional Hilbert space.
- The skew-adjoint structure implies conservation of the Hamiltonian.
- This structure is essential for the design of efficient and stable structure-preserving numerical schemes.

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