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Optimization problems in temporal graphs

Project Work I.

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1 Preliminaries and definitions

Computing a maximum matching in an undirected (static) graph is one of the most fundamental graph-algorithmic problems. It has many different variations and modifications, for example, it can also be extended for temporal graphs, whose topology is subject to discrete changes over time. In these graphs, edges can disappear and then reappear, allowing many real-world optimization problems to be modeled as a maximum matching problem in temporal graphs. However, even the maximum matching problem itself can be defined in various ways within these structures, which are in the focus of our work. In order to present our results precisely, we will need the following definitions.

Definition 1.1. *A **temporal graph** $\mathcal{G} = (G, \lambda)$ is a pair (G, λ), where $G = (V, E)$ is an underlying (static) graph and $\lambda : E \to 2^{\mathbb{N}} \setminus \{\emptyset\}$ is a **time-labeling function** that specifies which edge is active at what time, that is, an edge $e \in E$ is active at time $\tau \in \mathbb{N}$ if and only if $\tau \in \lambda(e)$. If e is active at time τ, then we call the pair (e, τ) a **time edge** and we say that e **appears** at time τ. The **lifetime** $\mathcal{T}(\mathcal{G}) < \infty$ of a temporal graph is the biggest time-label assigned to an edge, that is, $\mathcal{T}(\mathcal{G}) = \max\{\tau \in \lambda(e) \mid e \in E\}$.*

Definition 1.2. *Given a non-negative integer $\Delta \in \mathbb{N}$, we call two time edges (e, τ) and (e', τ') **Δ-independent** if the edges e, e' do not share an endpoint or their time-labels τ, τ' are at least Δ time units apart, that is, $|\tau - \tau'| \geq \Delta$. A **$\Delta$-matching** of a temporal graph $\mathcal{G}$ is a set M of pairwise Δ-independent time edges of $\mathcal{G}$.*

Problem 1.1. *In the **maximum Δ-matching problem**, the input is a temporal graph $\mathcal{G}$ and a non-negative integer $\Delta \in \mathbb{N}$. The goal is to find a Δ-matching in $\mathcal{G}$ with the largest possible cardinality.*

Definition 1.3. *Given a temporal graph $\mathcal{G} = (G, \lambda)$, $G = (V, E)$, an edge of the underlying graph $e \in E$ and a non-negative integer $\gamma \in \mathbb{N}$, the **γ-edge of e at time τ** is the set $\{(e, \tau') \mid \tau' \in \{\tau, \ldots, \tau + \gamma - 1\}\}$, where $\{\tau, \ldots, \tau + \gamma - 1\} \subseteq \lambda(e)$. We denote this by $\Gamma_\gamma(e, \tau)$. We say that two γ-edges $\Gamma_\gamma(e, \tau)$ and $\Gamma_\gamma(e', \tau')$ are **γ-independent** if the edges e, e' do not share an endpoint or $\{\tau, \ldots, \tau + \gamma - 1\}$ and $\{\tau', \ldots, \tau' + \gamma - 1\}$ are disjoint. A **γ-matching** of a temporal graph $\mathcal{G}$ is a set M of pairwise independent γ-edges.*

Problem 1.2. *In the **maximum γ-matching problem**, the input is a temporal graph $\mathcal{G}$ and a non-negative integer $\gamma \in \mathbb{N}$. The goal is to find a γ-matching in $\mathcal{G}$ with the largest possible cardinality.*

In my BSc thesis, we focused on the previous two matching problems: Δ- and γ-matching [1, 2]. Additionally, we explored how the seemingly unrelated d-matching problem [3, 4] connects to temporal graphs and to the two previous problems. We also analyzed the maximum Δ-matching problem in the case where the underlying static graph of the temporal graph is a tree and proved three results, which are as follows. We proved that this problem becomes NP-hard as soon as every edge can appear at most twice. We also proved that it can be solved in polynomial time if every edge appears at most once. Moreover, we gave an Efficient Polynomial-Time Approximation Scheme for the problem, provided that Δ is constant.

2 New results

In this semester, we continued examining the maximum Δ-matching problem on trees. Last year, we already established that if every edge appears at most once, then the problem can be solved in polynomial time. This raised the question of under what other conditions the problem is solvable. We managed to find several answers to this question, which are the following.

Theorem 2.1. *Consider the maximum Δ-matching problem, where the underlying graph $G = (V, E)$ of the input temporal graph $\mathcal{G} = (G, \lambda)$ is a tree. If there exists an optimal solution M^* for which the sets $M_v^* = \{(e, t) \in M^* \mid e \text{ is incident to } v\}$ are of constant size for every $v \in V$ and this constant K is known, then the problem can be solved in $O(n^2(K\mathcal{T}^2)^K)$ time.*

Corollary 2.1. *Maximum Δ-matching can be solved in $O(n^2K^{3K})$ time if the underlying graph $G = (V, E)$ of the input temporal graph $\mathcal{G} = (G, \lambda)$ is a tree, and $\mathcal{T}/\Delta \leq K$, where K is constant.*

Theorem 2.2. *Consider the maximum Δ-matching problem, where the underlying graph $G = (V, E)$ of the input temporal graph $\mathcal{G} = (G, \lambda)$ is a tree. Let N denote the size of the input ($N = O(|E|\mathcal{T}) = O(|V|\mathcal{T})$). If $\sum_{u\in\Gamma(v)} |\lambda(uv)| = O(\log N)$ for every $v \in V$, where $\Gamma(v)$ denotes the set of neighbors of v, then the problem can be solved in polynomial time.*

Using Corollary 2.1, we managed to generalize the EPTAS given in my BSc thesis. Let us note that here—unlike in the theorem stated in the thesis—we no longer assume that Δ is constant.

Theorem 2.3. *For every $0 < \varepsilon < 1$, the maximum Δ-matching problem admits an $O(\mathcal{T}n^2(1/\varepsilon)^{3/\varepsilon})$-time $(1 - \varepsilon)$-approximation algorithm if the underlying graph G of the input temporal graph $\mathcal{G}$ is a tree.*

Since the maximum γ- and the maximum d-matching problems can be reduced to the maximum Δ-matching problem, and this reduction is approximation-preserving, this result also applies to the other two problems. However, this is only interesting if those problems are also NP-hard on trees. There had been no results on this before, but during the semester, we proved that the maximum γ-matching problem is NP-hard on trees. In order to do this, we had to show that the maximum Δ-matching problem can be reduced to the maximum γ-matching problem in certain cases, which is also a new result (the reduction in the other direction was known). We also proved that the maximum d-matching problem is polynomial-time solvable on trees, therefore the EPTAS result is only interesting in the case of the maximum γ-matching problem.

In summary, during the semester we examined several new cases in which the maximum Δ-matching problem on trees is solvable. Using these results, we were able to strengthen the theorem presented in my thesis, and we now have a complete picture of the computational complexity of the maximum Δ-matching problem on trees. With the results achieved, we also obtained some new findings for the maximum γ- and d-matching problems. In the near future, we plan to publish a preprint on these results. We will continue the research on graphs with constant treewidth, as many results obtained for trees (especially those derived using dynamic programming) might also hold for such graphs.

References

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