Optimization problems in temporal graphs

Márk Hunor Juhász
Supervisor: Péter Madarasi

Project Work I.

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Definitions

Temporal graph

A temporal graph $\mathcal{G} = (G, \lambda)$ is a pair (G, λ), where $G = (V, E)$ is an underlying (static) graph and $\lambda : E \to 2^{\mathbb{N}} \setminus \{\emptyset\}$ is a time-labeling function that specifies which edge is active at what time, that is, an edge $e \in E$ is active at time $\tau \in \mathbb{N}$ if and only if $\tau \in \lambda(e)$. If e is active at time τ, then we call the pair (e, τ) a time edge and we say that e appears at time τ. The lifetime $\mathcal{T}(\mathcal{G}) < \infty$ of a temporal graph is the biggest time-label assigned to an edge, that is, $\mathcal{T}(\mathcal{G}) = \max\{\tau \in \lambda(e) \mid e \in E\}$.

Δ-matching

Given a non-negative integer $\Delta \in \mathbb{N}$, we call two time edges (e, τ) and (e', τ') Δ-independent if the edges e, e' do not share an endpoint or their time-labels τ, τ' are at least Δ time units apart, that is, $|\tau - \tau'| \geq \Delta$. A Δ-matching of a temporal graph $\mathcal{G}$ is a set M of pairwise Δ-independent time edges of $\mathcal{G}$.

Definitions

γ-matching

Given a temporal graph $\mathcal{G} = (G, \lambda),\ G = (V, E)$, an edge of the underlying graph $e \in E$ and a non-negative integer $\gamma \in \mathbb{N}$, the γ-edge of e at time τ is the set $\{(e, \tau') \mid \tau' \in \{\tau, \dots, \tau + \gamma - 1\}\}$, where $\{\tau, \dots, \tau + \gamma - 1\} \subseteq \lambda(e)$. We denote this by $\Gamma_\gamma(e, \tau)$. We say that two γ-edges $\Gamma_\gamma(e, \tau)$ and $\Gamma_\gamma(e', \tau')$ are γ-independent if the edges e, e' do not share an endpoint or $\{\tau, \dots, \tau + \gamma - 1\}$ and $\{\tau', \dots, \tau' + \gamma - 1\}$ are disjoint. A γ-matching of a temporal graph $\mathcal{G}$ is a set M of pairwise independent γ-edges.

d-matching

In the bipartite graph $G = (S, T, E)$, M is a d-matching if we obtain a matching by restricting M to any d consecutive nodes from S. Here, restricting M to a set of nodes $X \subseteq S$ means that we only consider the edges in M, which are induced by X and T.

Preliminaries

Theorem

The maximum Δ-matching problem is NP-hard even if $\Delta = 2$, the underlying graph of the input temporal graph is a tree and every edge appears at most twice.

Theorem

The maximum Δ-matching problem can be solved in polynomial time if the underlying graph $G = (V, E)$ of the input temporal graph $\mathcal{G} = (G, \lambda)$ is a tree, and every edge appears at most once.

Theorem

For every $0 < \varepsilon < 1$, the maximum Δ-matching problem admits an $O(\mathcal{T} n^2 (4\Delta/\varepsilon)^{\Delta/\varepsilon})$-time $(1 - \varepsilon)$-approximation algorithm if the underlying graph G of the input temporal graph $\mathcal{G}$ is a tree and $\Delta \in \mathbb{Z}_+$ is a positive constant.

New results: solvability

Theorem

Consider the maximum Δ-matching problem, where the underlying graph $G = (V, E)$ of the input temporal graph $\mathcal{G} = (G, \lambda)$ is a tree. If there exists an optimal solution M^* for which the sets $M_v^* = \{(e, t) \in M^* \mid e \text{ is incident to } v\}$ are of size at most K for every $v \in V$, then the problem can be solved in $O(n^2(K\mathcal{T}^2)^K)$ time.

Corollary

Maximum Δ-matching can be solved in $O(n^2 K^{3K})$ time if the underlying graph $G = (V, E)$ of the input temporal graph $\mathcal{G} = (G, \lambda)$ is a tree, and $\mathcal{T} \leq K$.

New results: solvability

Corollary

Maximum Δ-matching can be solved in $O(n^2K^{3K})$ time if the underlying graph $G = (V, E)$ of the input temporal graph $\mathcal{G} = (G, \lambda)$ is a tree, and $\mathcal{T}/\Delta \leq K$.

Theorem

Consider the maximum Δ-matching problem, where the underlying graph $G = (V, E)$ of the input temporal graph $\mathcal{G} = (G, \lambda)$ is a tree. Let N denote the size of the input ($N = O(|E|\mathcal{T}) = O(|V|\mathcal{T})$). If $\sum_{u \in \Gamma(v)} |\lambda(uv)| = O(\log N)$ for every $v \in V$, where $\Gamma(v)$ denotes the set of neighbors of v, then the problem can be solved in polynomial time.

New results: EPTAS

Theorem

For every $0 < \varepsilon < 1$, the maximum Δ-matching problem admits an $O(\mathcal{T} n^2 (1/\varepsilon)^{3/\varepsilon})$-time $(1-\varepsilon)$-approximation algorithm if the underlying graph G of the input temporal graph $\mathcal{G}$ is a tree.

Corollary

There is an EPTAS algorithm for the maximum γ-matching problem if the underlying graph G of the input temporal graph $\mathcal{G}$ is a tree.

New results: γ- and d-matchings

Theorem

Consider the maximum Δ-matching problem, where the underlying graph is $G = (V, E)$, the input temporal graph is $\mathcal{G} = (G, \lambda)$, and for every edge $e \in E$, $|x - y| > \Delta$ holds for every $x, y \in \lambda(e)$. In this case, the maximum Δ-matching problem can be reduced to the maximum γ-matching problem.

Corollary

The maximum γ-matching problem is NP-hard, even if the underlying graph of the temporal graph is a tree.

Theorem

The maximum d-matching problem can be solved in polynomial time if the input graph G is a tree.

Literature

- George B. Mertzios, Hendrik Molter, Rolf Niedermeier, Viktor Zamaraev, and Philipp Zschoche. „*Computing maximum matchings in temporal graphs*". In: *Journal of Computer and System Sciences* 137 (2023), pp. 1–19.
- Péter Madarasi. „*Matchings under distance constraints I.*" In: *Annals of Operations Research* 305 (2021), pp. 137–161.
- Péter Madarasi. „*Matchings under distance constraints II.*" In: *Annals of Operations Research* 332 (2024), pp. 303–327.

Where did I use AI?

I used ChatGPT to translate my Hungarian sentences to English.

Thank you for your attention!