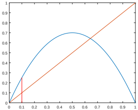
Picard-Kačanov-type iterations for nonlinear elliptic PDEs

Sike András

Project Work Presentation

Eötvös Loránd Tudományegyetem

January 8, 2026



Reminder

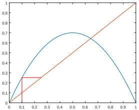
Last semester, we have investigated the following problem for $u : \Omega \to \mathbb{R}$:

$$\begin{cases} -\mathrm{div}(A(x,u)\cdot \nabla u) + q(x,u)u = f; \quad \text{in } \Omega \\ u\big|_{\Gamma_D} = 0; \\ (A(x,u)\nabla u \cdot \nu + pu)\big|_{\Gamma_N} = g, \end{cases}$$

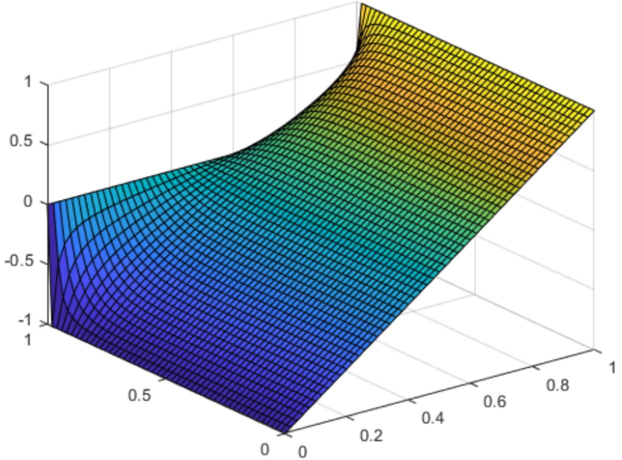
and introduced an iteration via

$$\begin{cases} -\mathrm{div}(A(x,u^{(n)})\cdot \nabla u^{(n+1)}) + q(x,u^{(n)})u^{(n+1)} = f; \quad \text{in } \Omega \\ u^{(n+1)}\big|_{\Gamma_D} = 0; \\ (A(x,u^{(n)})\nabla u^{(n+1)} \cdot \nu + pu^{(n+1)})\big|_{\Gamma_N} = g. \end{cases}$$

This leads to a series of linear PDEs, which can be solved numerically.



SDFEM1, ε=1/10



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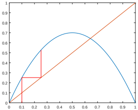
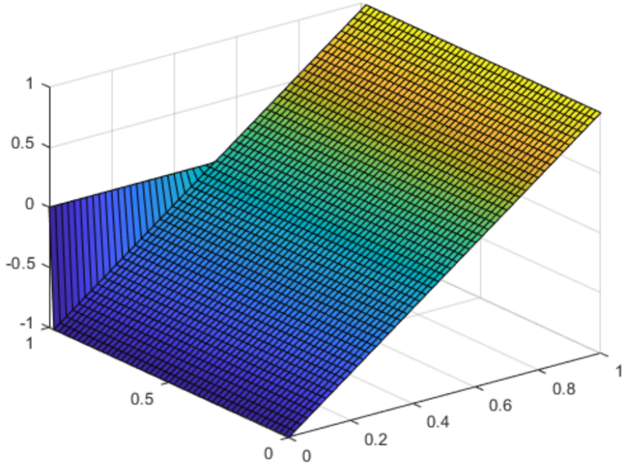
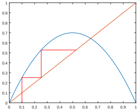
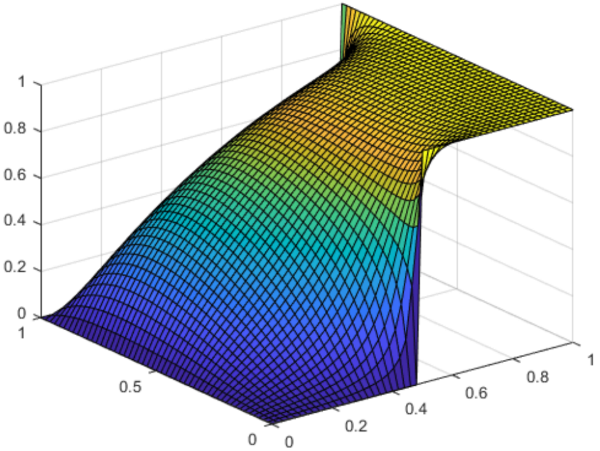


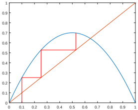
SDFEM1, ε=1/100



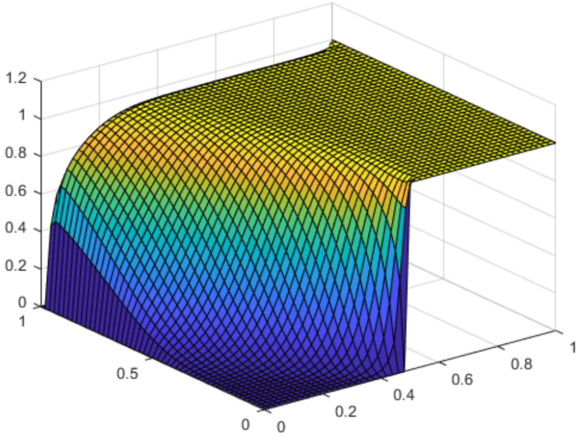


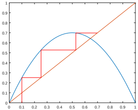
SDFEM2, ε=1/10





SDFEM2, ε=1/100





FEM for the Linearized PDE

$$\begin{cases} -\mathrm{div}(A(x,u^{(n)}) \cdot \nabla u^{(n+1)}) + q(x,u^{(n)})u^{(n+1)} = f; \quad \text{in } \Omega \\ u^{(n+1)}\big|_{\Gamma_D} = 0; \\ (A(x,u^{(n)})\nabla u^{(n+1)} \cdot \nu + pu^{(n+1)})\big|_{\Gamma_N} = g. \end{cases}$$

Take the weak formulation

$$\int_\Omega A(x,u^{(n)}) \cdot \nabla u^{(n+1)} \cdot \nabla v + \int_\Omega q(x,u^{(n)})u^{(n+1)}v = \int_\Omega fv + \int_{\partial\Omega} pv \qquad \forall\, v \in H_D^1(\Omega),$$

Then some subspace $V_h \subset H_D^1(\Omega)$, and define for the basis functions $\varphi_j, \varphi_i \in V_h$ the matrix

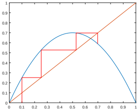
$$A_n := \int_\Omega A(x,u^{(n)}) \cdot \nabla\varphi_j \cdot \nabla\varphi_i + \int_\Omega q(x,u^{(n)})\varphi_j\varphi_i$$

and the vector

$$b_n := \int_\Omega f\varphi_i + \int_{\partial\Omega} p\varphi_i.$$

Then we just have to solve the linear equation

$$A_n c = b_n.$$



Nonnegativity Principle for a Basic Domain

$$\begin{cases} \partial_t u - \operatorname{div}\left(A(u)\nabla u\right) = f & \text{in } \Omega \\ u(x,0) = 0 & \forall x \in \Omega, \\ u\big|_{\Gamma_D \times \mathbb{R}^+} = 0, & \\ A(u)\partial_\nu u + \alpha(u - u_0)\big|_{\Gamma_N \times \mathbb{R}^+} = 0. & \end{cases}$$

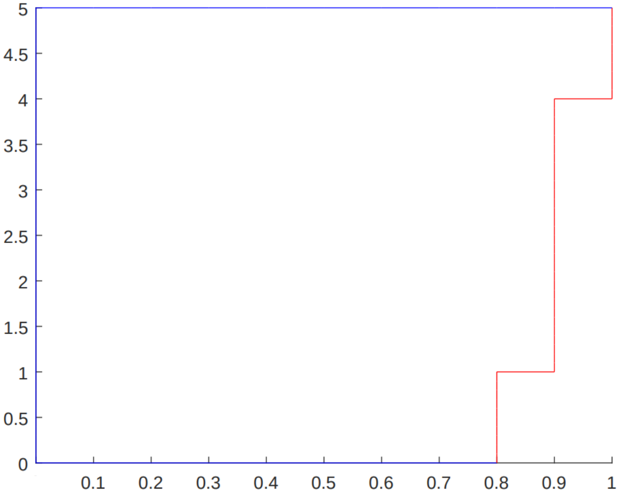
First, we describe time layers $t_j = j \cdot \tau$. We freeze the relevant coefficents, and apply a finite difference scheme for the time derivative:

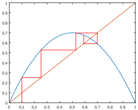
$$\frac{1}{\tau}(u^{(j+1)} - u^{(j)}) - \operatorname{div}\left(A(u^{(j)})\nabla u^{(j+1)}\right) = f,$$

so the weak formulation is

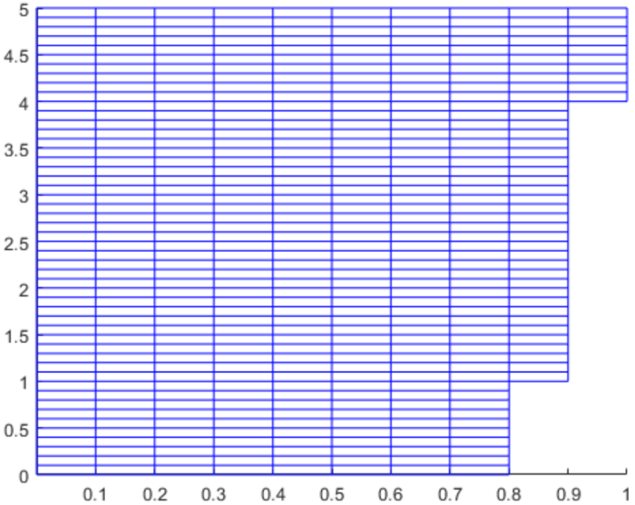
$$\int_\Omega A(u^{(j)}) \cdot \nabla u^{j+1} \cdot \nabla v + \int_\Omega \frac{1}{\tau} u^{(j+1)} v + \int_{\Gamma_N} \alpha u^{(j+1)} v = \int_\Omega f v + \int_\Omega \frac{1}{\tau} u^{(j)} v + \int_{\Gamma_N} u_0 v.$$

The Domain with Boundary Conditions





The Discretized Domain



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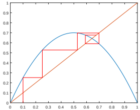


Discrete Nonnegativity Principle

How can we guarantee that the nonnegativity principle holds for the numerical solution, and how practical are the conditions?
If we are given a time step τ, then we have to have a mesh with mesh size of at most

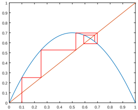
$$h_{\mathrm{opt}} = \frac{\mu_0}{\frac{\alpha}{2} + \sqrt{\left(\frac{\alpha}{2}\right)^2 + \frac{\mu_0}{3\tau}}},$$

conversely, if h mesh size is given, then

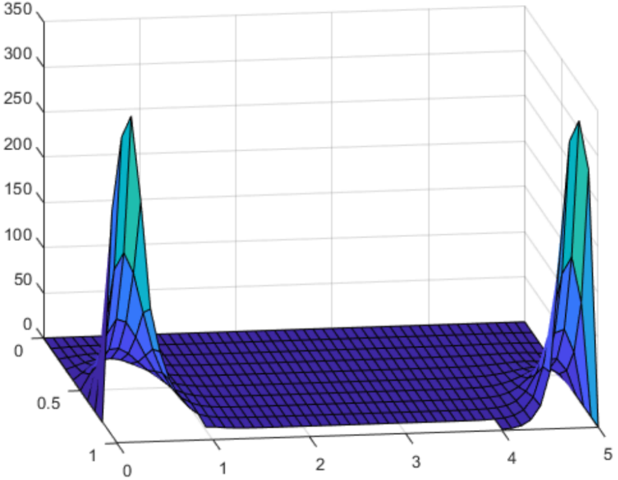
$$\tau_{\mathrm{opt}} := \frac{1}{\frac{3}{\mu_0}\left(\left(\frac{h}{\mu_0} - \frac{\alpha}{2}\right)^2 - \left(\frac{\alpha}{2}\right)^2\right)}$$

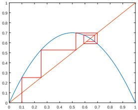
is the theoretical minimum temporal step.

This is not a condition that is only theoretically interesting!

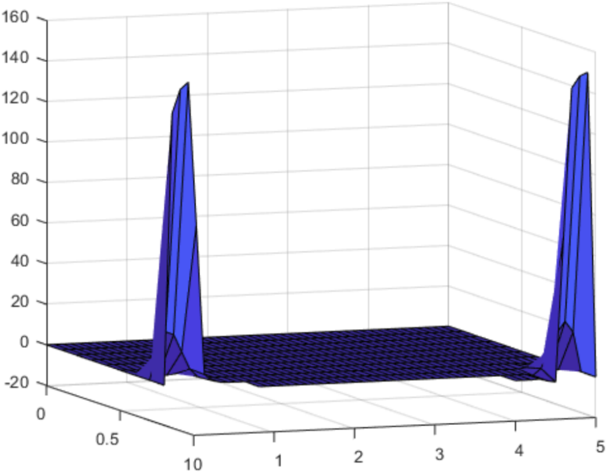


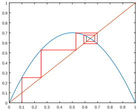
Solution, $h = 0.1$, $\tau = 0.01$



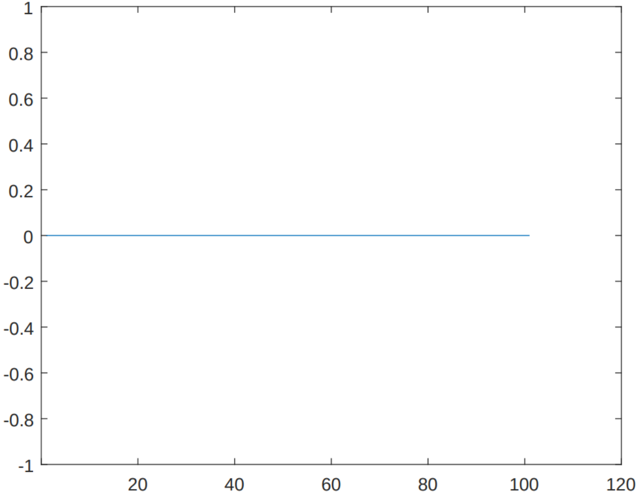


Solution, $h = 0.1$, $\tau = 0.001$

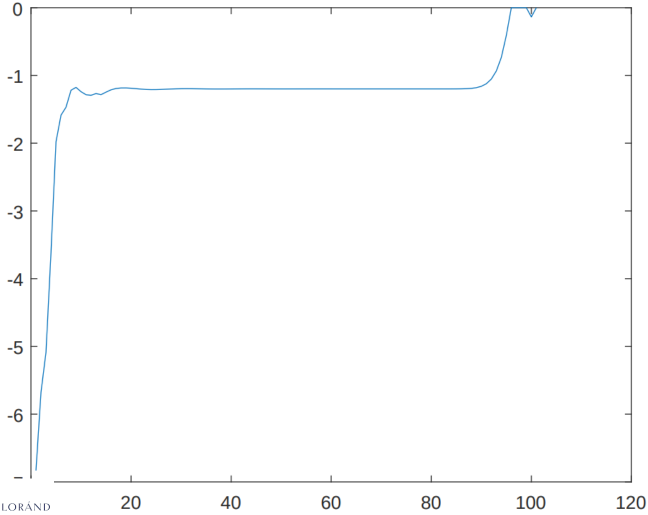




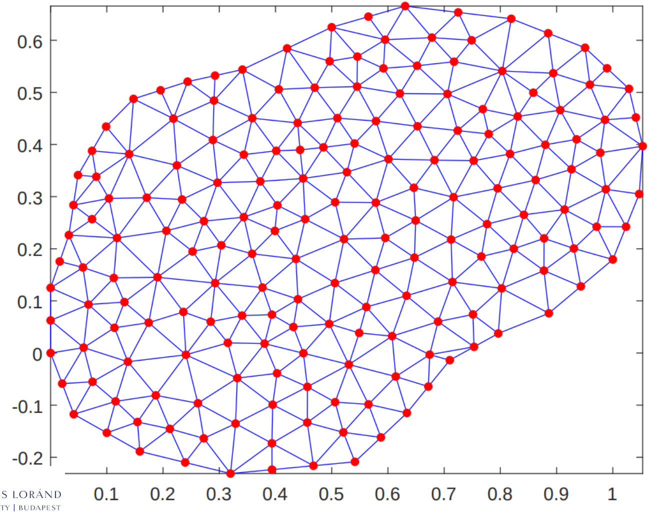
Evolution of the errors, $h = 0.1$, $\tau = 0.01$



Evolution of the errors, $h = 0.1$, $\tau = 0.001$



Coarse Discretization of an Arbitrary Domain



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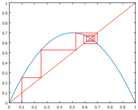
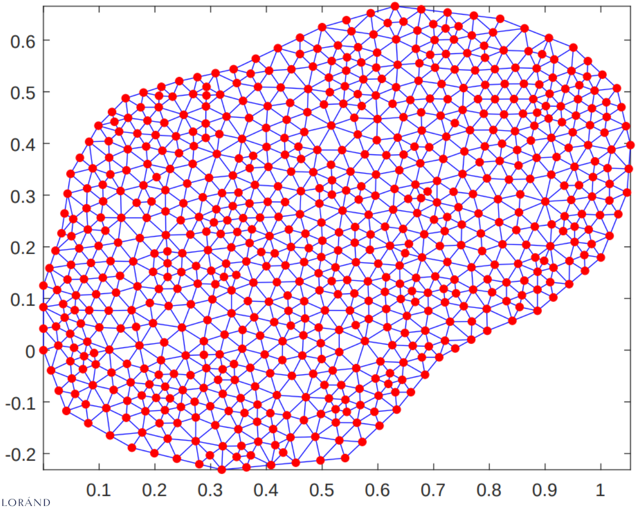
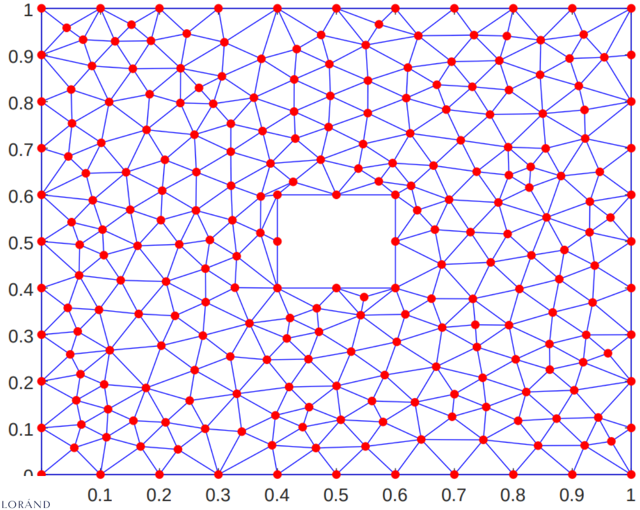
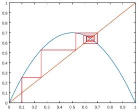


Fine Discretization of an Arbitrary Domain

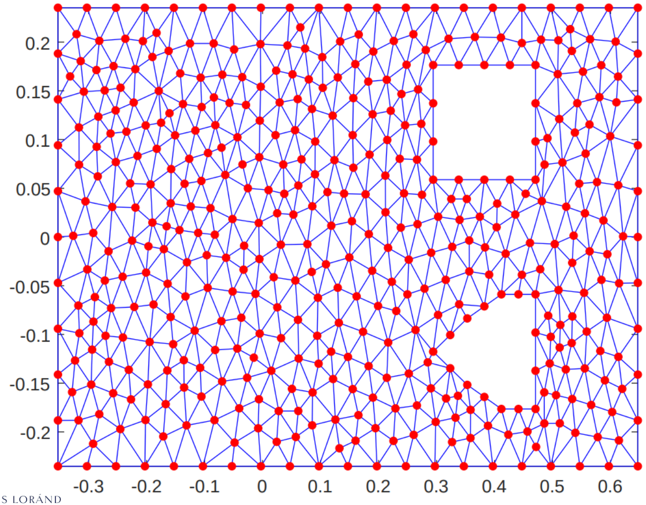


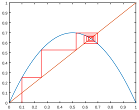
Domain with a Hole





Wind Tunnel with Objects





Michaelis-Menten Equation in a Cell

$$
\begin{cases}
\partial_t u - \mu_0 \Delta u + \dfrac{u}{1+\varepsilon u} = f & u : \Omega \times \mathbb{R}^+ \to \mathbb{R}, \\
u(x,0) = 0 & \forall\ x \in \Omega, \\
u\big|_{\partial\Omega \times \mathbb{R}^+} = 0. &
\end{cases}
$$

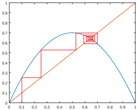
Here, we have the estimates

$$
\tau_{\text{opt}} := \frac{1}{\frac{12\cos(\alpha_0)\cdot\mu_0}{h_0} - 1},
$$

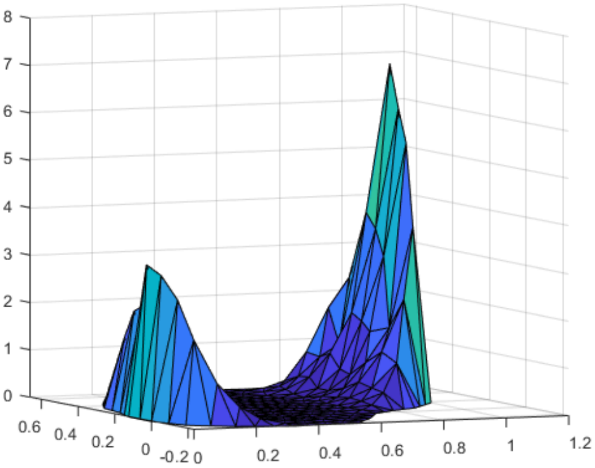
and

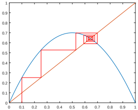
$$
h_{\text{opt}} := \left(\frac{12\cos(\alpha_0)\cdot\mu_0}{1+\frac{1}{\tau}} \right)^{1/2},
$$

where h_0 and α_0 denote the largest side and angle in the triangulation respectively.

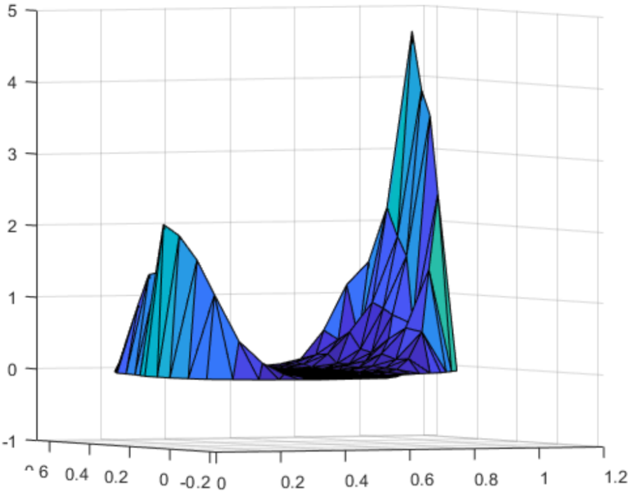


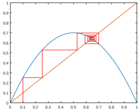
Solution with $h = 0.06$, $\tau = 0.2$



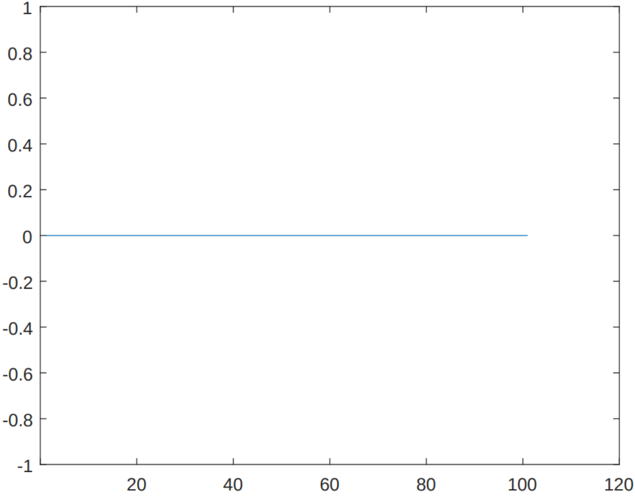


Solution with $h = 0.06$, $\tau = 0.1$





Evolution of the Errors, $h = 0.06$, $\tau = 0.2$



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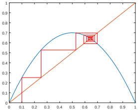
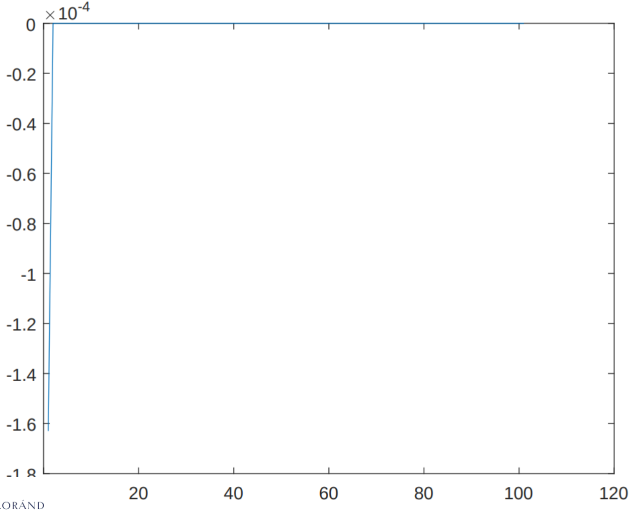


Evolution of the Errors, $h = 0.06$, $\tau = 0.1$



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