Searching and generating sparse (sub)graphs

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Project work III.
8th January, 2026

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Example (Laman, 1970)

G is minimally rigid in the plane $\Leftrightarrow$ G is $(2, 3)$-tight.

Problem (maximum-weight (k, ℓ)-sparse subgraph)

Input: *A graph* $G = (V, E)$ *with a weight function* $w : E \to \mathbb{R}_+$.

Output: $F \subseteq E$ *such that* $H = (V, F)$ *is sparse and* $w(F) = \sum_{e \in F} w(e)$ *is maximal.*

The considered problems

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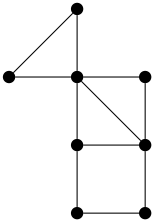
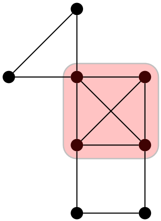
Problem (recognizing (k, ℓ)-sparse graphs)

Input: *A graph* $G = (V, E)$.

Output: `true` *if* G *is sparse, otherwise* `false`.



A $(2, 3)$-sparse graph



A graph that is not $(2, 3)$-sparse

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Now all of these can be solved in quadratic time.

Recognition – Intro

Note

When $0 \leq \ell < 2k$, the recognition algorithm is a matroid independence oracle.

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$0 \le \ell \le k$	general case	$O(n\sqrt{n \log n})$	(Gabow, Westermann, 1988)
	case $\ell = k$	$O(n^{1+o(1)})$	(Arkhipov, Kolmogorov, 2024)
$k < \ell < 2k$	general case	$O(n^2)$	(Berg, Jordán, 2003; Lee, Streinu, 2008)
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An open question from rigidity theory:

Question

Is there a subquadratic algorithm for recognizing $(2,3)$-sparse graphs?

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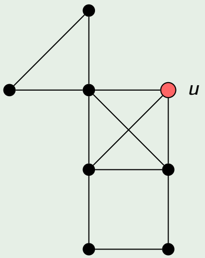
Let $(k, \ell) = (2, 3)$, $U_0 = \{u\}$.



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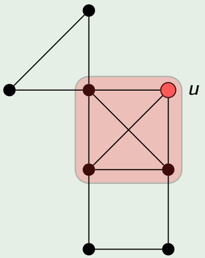
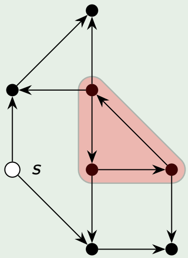
Let $(k, \ell) = (2, 3)$, $U_0 = \{u\}$. Consider the graph G and digraph D below. Here, $X \ni u$ violates the sparsity of $G \Leftrightarrow X - u$ violates rooted 1-arc-connectivity in D.





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Corollary

It suffices to detect if there is a violating set saturated by a given spanning tree F.

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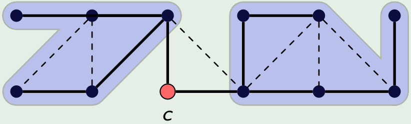

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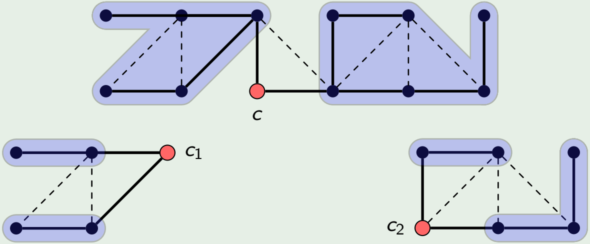

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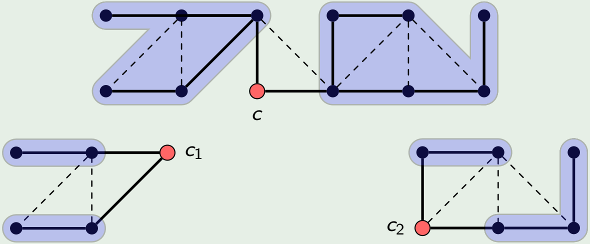

Note

For the Laman case, we get the best previous running time, but our algorithm is simpler, much more general, and also provides a violating set.

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The range $2k \leq \ell < 3k$:

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2. Edge uv can be inserted $\Leftrightarrow$ there is no $X \supsetneq \{u, v\}$ violating $(k, \ell + 1)$-sparsity.
3. Apply the reduction for $U_0 = \{u, v\}$.

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Our papers:

- Optimization problem: Quadratic-Time Algorithm for the Maximum-Weight (k, ℓ)-Sparse Subgraph Problem (https://arxiv.org/abs/2511.20882)
- Recognition problem: in progress...

Throughout the project, no AI/ML tools were used.

Thank you for your attention!