Directed Studies Report

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May 18, 2025

In my directed study, I began exploring the theory of infinite games, with particular focus on their connections to large cardinals. It is precisely this connection that I find especially interesting, as the existence of winning strategies for games played on $\mathbb{N}$ or $\mathbb{R}$ is closely related to the consistency of certain large cardinals high in the hierarchy of infinities.
I based my work on [1].

Definition. Let X be a non-empty set and $A \subseteq {}^{\omega}X$. The game $G_X(A)$ is played between two players, I and II, who alternately choose elements of X to form an infinite sequence $x \in {}^{\omega}X$:

$$\begin{aligned} \mathrm{I} &: x(0),\ x(2),\ x(4),\ \ldots \\ \mathrm{II} &: x(1),\ x(3),\ x(5),\ \ldots \end{aligned}$$

Player I wins if $x \in A$; otherwise, player II wins.
A strategy for player I is a function $\sigma\colon \bigcup_{n<\omega} X^{2n} \to X$ assigning I's move based on all previous moves. For $y \in {}^{\omega}X$, the *play according to* σ is the sequence $\sigma * y \in {}^{\omega}X$ defined recursively by:

$$(\sigma * y)(2n) = \sigma(y(0), y(1), \ldots, y(2n-1)), \quad (\sigma * y)(2n+1) = y(n).$$

Similarly, a strategy for II is a function $\tau\colon \bigcup_{n<\omega} X^{2n+1} \to X$. For $z \in {}^{\omega}X$, the sequence $z * \tau \in {}^{\omega}X$ is defined by:

$$(z * \tau)(2n) = z(n), \quad (z * \tau)(2n+1) = \tau(z(0), z(1), \ldots, z(n)).$$

A strategy σ for I is *winning* if $\sigma * y \in A$ for all $y \in {}^{\omega}X$; similarly, τ is a winning strategy for II if $z * \tau \notin A$ for all $z \in {}^{\omega}X$.
The game $G_X(A)$ is called *determined* if one of the players has a winning strategy.

Theorem (Gale–Stewart)**.** Assuming the Axiom of Choice, it follows that there exists a subset of the reals which is not determined.

Fortunately, in set theory, the absence of a result under the Axiom of Choice does not mark the end of investigation. We are often able - and sometimes even prefer - to work without the axiom.
I begin by presenting three games whose associated winning strategies yield clear characterizations of certain regularity properties of subsets of the real line.

Definition (Banach-Mazur Game)**.** Let $A \subseteq {}^{\omega}\omega$. The game $G^{**}_{\omega}(A)$ is played by two players I and II who alternately choose non-empty finite sequences of natural numbers. The outcome is the concatenation $x = s_0{}^\frown s_1{}^\frown s_2{}^\frown \cdots$. Player I wins if $x \in A$; otherwise, player II wins.

Proposition (Mazur, Banach–Mauldin, Oxtoby)**.** Let $A \subseteq {}^{\omega}\omega$.

(a) A is meager if and only if player II has a winning strategy in $G^{**}_{\omega}(A)$.

(b) There exists $s \in {}^{<\omega}\omega$ such that $O(s) \setminus A$ is meager if and only if player I has a winning strategy in $G^{**}_{\omega}(A)$.

Corollary. Let $A \subseteq {}^{\omega}\omega$, and define

$$O_A = \bigcup \{O(s) \mid s \in {}^{<\omega}\omega,\ O(s) \setminus A \text{ is meager}\}.$$

If the game $G^{**}_{\omega}(A \setminus O_A)$ is determined, then A has the Baire property.

Definition (Perfect Set Game)**.** Let $A \subseteq {}^{\omega}2$. The game $G^{*}_{2}(A)$ is defined as follows: player I selects elements from ${}^{<\omega}2$ (finite binary sequences), while player II chooses elements from $2 = \{0, 1\}$. The resulting sequence is

$$x = s_0\hat{\ }k_1\hat{\ }s_2\hat{\ }k_3 \cdots$$

and player I wins if $x \in A$; otherwise, player II wins.

Proposition (Davis)**.** Let $A \subseteq {}^{\omega}2$.

(a) A is countable if and only if player II has a winning strategy in $G^{*}_{2}(A)$.

(b) A has a perfect subset if and only if player I has a winning strategy in $G^{*}_{2}(A)$.

Remark. With a suitable technical encoding, this game can be transformed into one played on the real numbers.

Definition (Minimal Cover)**.** Let $A \subseteq {}^{\omega}\omega$. A set $B \subseteq {}^{\omega}\omega$ is a *minimal cover* for A if the following hold:

- $A \subseteq B$,
- B is Lebesgue measurable,
- for every $Z \subseteq B \setminus A$ with Z Lebesgue measurable, we have $m_L(Z) = 0$.

Definition (Covering Game)**.** Let $A \subseteq {}^{\omega}\omega$ and let $\varepsilon > 0$. The game $G(A, \varepsilon)$ is played between two players I and II who alternately choose natural numbers to form a sequence $x \in {}^{\omega}\omega$:

$$\begin{array}{ll} \text{I:} & x(0),\ x(2),\ x(4),\ \ldots \\ \text{II:} & x(1),\ x(3),\ x(5),\ \ldots \end{array}$$

We assume a fixed recursive surjection $\Psi : {}^{\omega}\omega \to {}^{\omega}2$, and let Ψ^{-1} denote its inverse on the image.

The sequence $x_I = (x(0), x(2), \ldots)$ must consist of 0s and 1s and must not be eventually constant. Each of II's moves codes a finite union of basic open sets. Using a recursive enumeration $(s_i)_{i\in\omega}$ of ${}^{<\omega}\omega$, let:

$$N_i = O(s_{i(0)}) \cup \cdots \cup O(s_{i(|i|-1)}).$$

Each move $x(2n + 1)$ by II must satisfy

$$m_L(N_{x(2n+1)}) < \frac{\varepsilon}{2^{2(n+1)}}.$$

Player I wins if

$$\Psi^{-1}(x_I) \in A \setminus \bigcup_{n<\omega} N_{x(2n+1)};$$

otherwise, player II wins.

Proposition. In the game $G(A, \varepsilon)$, the following hold:

(a) If player I has a winning strategy, then there exists a Lebesgue measurable set $B \subseteq A$ with $m_L(B) > 0$.

(b) If player II has a winning strategy, then there exists an open set $O \supseteq A$ such that $m_L(O) < \varepsilon$.

Corollary. Suppose $A \subseteq {}^\omega\omega$, and $B \subseteq {}^\omega\omega$ is a minimal cover for A. If $\epsilon > 0$, then the game $G(B \setminus A, \epsilon)$ is determined. Consequently, A is Lebesgue measurable.

We have now arrived at our central concept: the Axiom of Determinacy.

Definition (Axiom of Determinacy (AD))**.** Every set of reals is determined.

Here, the reals are identified with the Baire space ${}^\omega\omega$, and "determined" means that the game $G_\omega(A)$, played on ${}^\omega\omega$ as described earlier, has a winning strategy for one of the players for every $A \subseteq {}^\omega\omega$.

Definition (Real Determinacy ($\mathrm{AD}_\mathbb{R}$))**.** For every $A \subseteq {}^\omega({}^\omega\omega)$, the game $G_\omega(A)$ is determined.

In this stronger form, the determinacy extends to games where the players choose real numbers (elements of ${}^\omega\omega$) at each move, so the entire play forms a sequence of reals — an element of ${}^\omega({}^\omega\omega)$.

Theorem (Mycielski–Świerczkowski; Mazur, Banach; Davis)**.** Assume AD. Then every set of reals is Lebesgue measurable, has the Baire property, and has the perfect set property.

Although the Axiom of Determinacy (AD) is incompatible with the Axiom of Choice (AC), it implies a weaker form of choice.

Definition. The principle $\mathrm{AC}_\omega({}^\omega\omega)$ asserts that every countable set of non-empty subsets of ${}^\omega\omega$ (of reals) has a choice function.

Proposition (Świerczkowski, Mycielski, Scott)**.** The Axiom of Determinacy implies $\mathrm{AC}_\omega({}^\omega\omega)$, and hence ω_1 is regular.

Definition ($\mathrm{AC}_\omega({}^\omega\omega)$)**.** The axiom $\mathrm{AC}_\omega({}^\omega\omega)$ asserts that every countable set of non-empty subsets of ${}^\omega\omega$ (i.e., of reals) admits a choice function.

Proposition (Świerczkowski, Mycielski, Scott)**.** The Axiom of Determinacy implies $\mathrm{AC}_\omega({}^\omega\omega)$, and hence ω_1 is regular.

The following lemmas provide a foundation for deriving new and interesting results in subsequent investigations.

Lemma (ZF)**.** If ω_1 is regular and $\forall a \in {}^\omega\omega(\omega_1^{L[a]} < \omega_1)$, then

$$\forall a \in {}^\omega\omega(\omega_1 \text{ is inaccessible in } L[a]).$$

Lemma (ZF, Bernstein)**.** The following statements hold:

(a) If the reals are well-orderable, then there exists a set A of reals with cardinality $2^{\aleph_0}$ such that for every perfect set P of reals, both $P \cap A \neq \emptyset$ and $P \setminus A \neq \emptyset$.

(b) If $\omega_1 \leq 2^{\aleph_0}$, then there exists a set of reals that does not have the perfect set property.

Lemma (Solovay, Specker, Levy)**.** The following theories are equiconsistent:

(a) $\mathrm{ZFC} + \exists\kappa\,(\kappa \text{ is inaccessible})$.

(b) $\mathrm{ZF} + \mathrm{DC} +$ "every set of reals has the perfect set property".

(c) $\mathrm{ZF} + \omega_1$ is regular $+\, \omega_1 \not\leq 2^{\aleph_0}$.

(d) $\mathrm{ZF} + \omega_1$ is regular $+\, \forall a \in {}^\omega\omega(\omega_1^{L[a]} < \omega_1)$.

The Axiom of Determinacy (AD) has deep consequences for the structure of the real line and its subsets. Among them are strong limitations on how large sets of reals can be organized:

Proposition (Mycielski)**.** The following hold assuming AD:

(a) $\omega_1 \not\leq 2^{\aleph_0}$, i.e., there is no uncountable well-orderable set of reals.

(b) $\mathrm{Con}(\mathrm{ZF} + \mathrm{AD}) \Rightarrow \mathrm{Con}(\mathrm{ZFC} + \exists\kappa\,(\kappa \text{ inaccessible}))$.

Definition. Let $\mathcal{A}$ be a class of sets of reals. Then

$$\mathrm{Det}(\mathcal{A}) \quad \text{means that every set of reals in } \mathcal{A} \text{ is determined.}$$

A particularly important instance of this schema is *Projective Determinacy*, abbreviated as PD, which asserts that every projective set of reals is determined. The projective sets are those obtainable from the Borel sets by a finite number of applications of projection and complementation, forming the hierarchy

$$\Sigma^1_1,\ \Pi^1_1,\ \Sigma^1_2,\ \Pi^1_2,\ \ldots$$

Thus, Projective Determinacy is the statement:

$$\mathrm{PD} := \mathrm{Det}\left(\bigcup_{n<\omega} \Pi^1_n\right).$$

In the following, I list some interesting results from the topic, without aiming at completeness.

Proposition ($\mathrm{AC}_\omega({}^\omega\omega)$)**.**

$$\mathrm{Det}(\Pi^1_n) \Rightarrow \forall a \in {}^\omega\omega,\ \omega_1 \text{ is inaccessible in } L[a].$$

This expresses that under projective determinacy, inner models $L[a]$ look quite canonical from the standpoint of admissibility: ω_1 appears as a large cardinal in each such model.

Definition (Uniformization)**.** Let $A \subseteq X \times Y$. A function $f\colon X \to Y$ *uniformizes* A if

$$\forall x \in \mathrm{dom}(A)\ \ ((x, f(x)) \in A)\,,$$

f selects exactly one y for each x such that $(x, y) \in A$.

Uniformization ensures definable selections from relations, and is central in connections between determinacy and definability.

Theorem (Kechris, Martin)**.** Assume $\mathrm{AC}_\omega({}^\omega\omega)$ and $\mathrm{Det}(\Pi^1_n)$. Then every Σ^1_{n+1} set of reals is Lebesgue measurable, has the Baire property, and has the perfect set property.

This highlights the deep regularity consequences of determinacy: the sets exhibit regular and well-structured behavior in terms of classical descriptive set theory.

Proposition. Assume $\mathrm{AD}_\mathbb{R}$. Then every subset of $2^{({}^\omega\omega)}$ can be uniformized.

Proposition (Solovay)**.** Assume $V = L(\mathbb{R})$ and there is no well-ordering of the reals. Then there exists a subset of $2^{({}^\omega\omega)}$ that cannot be uniformized.

Together, these two results show that the possibility of uniformizing all subsets of ${}^\omega\omega$ is sensitive to determinacy versus the existence of a definable global well-order.

Proposition (Solovay)**.** Assume $\mathrm{AD}_\mathbb{R}$. Then for any $S \subseteq {}^\omega\omega$, we have

$$V \neq L(\mathbb{R} \cup \{S\}).$$

Under $\mathrm{AD}_\mathbb{R}$, the universe cannot be generated as a constructible extension of the reals and an arbitrary set of reals. This reflects the incompatibility of $\mathrm{AD}_\mathbb{R}$ with the existence of a definable inner model containing all reals.

Theorem (Solovay)**.** Assume AD. Then both ω_1 and ω_2 are measurable cardinals.

This result shows that under determinacy, certain small uncountable cardinals (in particular, ω_1 and ω_2) acquire large cardinal properties. Measurability here refers to the existence of a nonprincipal, ω-complete ultrafilter, which is a strong combinatorial feature typically associated with very large cardinals in ZFC.

Corollary. Assume AD. Then $\mathrm{cf}(\omega_n) = \omega_2$ for every $2 \leq n < \omega$.

Theorem (Kechris)**.** Assume AD and $V = L(\mathbb{R})$. Then DC holds.

This axiom is a weaker version of the Axiom of Choice and suffices for most of classical analysis. It states that for any nonempty set X and binary relation R on X, if for every $x \in X$ there exists $y \in X$ such that xRy, then there exists a sequence $(x_n)_{n<\omega}$ in X such that $x_n R x_{n+1}$ for all n.
This result shows that even though AD contradicts the Axiom of Choice in full generality, under the assumption that all sets are constructible from the reals ($V = L(\mathbb{R})$), a significant fragment of choice - the Axiom of Dependent Choices - still holds.

Corollary. If $n \in \omega$ and there exist n Woodin cardinals with a measurable cardinal above them, then

$$\mathrm{Det}(\Pi^1_{n+1}).$$

Corollary. If there are infinitely many Woodin cardinals with a measurable cardinal above them, then

$$\mathrm{AD}^{L(\mathbb{R})}.$$

These results show how large cardinal assumptions lead to determinacy for definable classes of sets. In particular, the existence of finitely or infinitely many Woodin cardinals (with a measurable above) guarantees determinacy at higher levels of the projective hierarchy and even full determinacy for all sets of reals in $L(\mathbb{R})$.

Theorem (Woodin)**.** The following theories are equiconsistent:

(a) ZFC+ There are infinitely many Woodin cardinals.

(b) ZF + AD.

Theorem (Woodin)**.** The following theories are equiconsistent:

(a) $\mathrm{ZFC} + \mathrm{Det}(\Pi^1_2)$.

(b) ZFC+ There exists a Woodin cardinal.

Theorem (Woodin)**.** The following are equivalent:

(a) $\mathrm{Det}(\Pi^1_2)$.

(b) For every $a \in {}^\omega\omega$, there exists a countable ordinal δ such that δ is Woodin in an inner model of ZFC containing a.

This highlights the remarkable connection between definable determinacy (like Π^1_2 determinacy) and the existence of large cardinals in suitable inner models. It underscores how determinacy axioms reflect the internal structure and consistency strength of strong set-theoretic assumptions.

Theorem (Woodin)**.** The following theories are equiconsistent:

(a) ZFC+ There are infinitely many Woodin cardinals.

(b) ZF + AD.

This result demonstrates that the assumption of infinite many Woodin cardinals has the same consistency strength as assuming the Axiom of Determinacy. In other words, while AD contradicts the Axiom of Choice, its logical consistency can be matched by a theory that remains within the framework of ZFC extended by large cardinal assumptions.

Theorem (Woodin)**.** The following theories are equiconsistent:

(a) $\mathrm{ZFC} + \mathrm{Det}(\Pi^1_2)$.

(b) ZFC+ There exists a Woodin cardinal.

This highlights that even a single Woodin cardinal suffices to account for the determinacy of all Π^1_2 sets of reals. Since such determinacy principles have profound implications in descriptive set theory (for example regularity properties, uniformization), the result emphasizes the strength of Woodin cardinals in foundational mathematics.

Theorem (Woodin)**.** The following are equivalent:

(a) $\mathrm{Det}(\Pi^1_2)$.

(b) For every $a \in {}^\omega\omega$, there exists a countable ordinal δ such that δ is Woodin in an inner model of ZFC containing a.

This equivalence establishes a deep bridge between projective determinacy and the inner model theory of large cardinals. It shows that determinacy for the class Π^1_2 is not only implied by large cardinal axioms, but is also intimately tied to their structural presence inside the constructible universe relative to any real parameter.

References

[1] Akihiro Kanamori, The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings, 2nd edition, Springer-Verlag, Berlin, 2003