Infinite Games and the Axiom of Determinacy

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Infinite Game, $G_X(A)$

Definition

Let X be a non-empty set, and let $A \subseteq {}^\omega X$, that is, a set consisting of infinite sequences over X.
The game $G_X(A)$ is played between two players, I and II, who take turns choosing elements from X:

$$\begin{array}{rl} \text{I:} & x(0),\ x(2),\ x(4),\ \ldots \\ \text{II:} & x(1),\ x(3),\ x(5),\ \ldots \end{array}$$

The result is an infinite sequence $x \in {}^\omega X$ (a function from the natural numbers to X).
Player I wins if $x \in A$; otherwise, player II wins.

Determinacy

Definition

A game $G_X(A)$ is called **determined** if one of the players has a winning strategy - that is, a function that guarantees victory regardless of the opponent's moves.

Remarks

- It is a natural and deep question to ask which games are determined and for which player.
- Infinite games can characterize important regularity properties (measurability, Baire, perfect set).
- In most applications, X is either ω or $2 = \{0, 1\}$, so the games take place over the reals: outcomes $x \in {}^{\omega}\omega$ or $x \in {}^{\omega}2$ naturally encode real numbers, so we can think of A as a set of real numbers.

Banach–Mazur Game, $G^{**}_{\omega}(A)$

Setup

Let $A \subseteq {}^{\omega}\omega$. Players I and II alternately choose non-empty finite sequences of natural numbers:

$$x = s_0{}^\frown s_1{}^\frown s_2{}^\frown \cdots$$

The outcome is $x \in {}^{\omega}\omega$, the infinite concatenation (writing the finite sequences one after another).
Player I wins if $x \in A$, otherwise Player II wins.

Theorem

- *A is meager (countable union of nowhere dense sets) $\iff$ Player II has a winning strategy.*
- *If the game $G^{**}_{\omega}(A \setminus O_A)$ is determined, then A has the Baire property, where O_A is the largest open set such that $O_A \setminus A$ is meager.*

Perfect Set Game, $G_2^*(A)$

Setup
Let $A \subseteq {}^{\omega}2$. Player I chooses finite binary sequences s_i, while Player II picks single bits k_i. The resulting sequence is:

$$x = s_0 \hat{} k_1 \hat{} s_2 \hat{} k_3 \cdots$$

Player I wins if $x \in A$, otherwise Player II wins.

Theorem

- *A is countable* $\iff$ *Player II has a winning strategy.*
- *A has a perfect subset (a nonempty closed set with no isolated points)* $\iff$ *Player I has a winning strategy.*

Covering Game, $G(A, \varepsilon)$

Setup

- Fix $A \subseteq {}^{\omega}\omega$, $\varepsilon > 0$.
- Players I and II alternate choosing natural numbers.
- From this point on, the definition becomes technical — we won't go into the details now.

Theorem

A set $A \subseteq {}^{\omega}\omega$ is Lebesgue measurable if for some minimal cover $B \supseteq A$, the game $G(B \setminus A, \varepsilon)$ is determined for some $\varepsilon > 0$.

Axiom of Determinacy

Definition

The Axiom of Determinacy (AD): For every $A \subseteq {}^{\omega}\omega$, $G_{\omega}(A)$ is determined.

Theorem

AD contradicts the Axiom of Choice (AC), but it is consistent with Zermelo–Fraenkel set theory (ZF).

Remark

- AD implies useful fragments like $\mathrm{AC}_{\omega}({}^{\omega}\omega)$ (every countable collection of nonempty subsets of ${}^{\omega}\omega$ has a choice function).
- In $\mathrm{ZF} + \mathrm{AD}$, all sets of reals have strong regularity properties.
- AD is closely related to the existence of large cardinals (measurable, Woodin).

Strong Cardinals under AD

Theorem

$\mathrm{Con}(\mathrm{ZF} + \mathrm{AD}) \Rightarrow \mathrm{Con}(\mathrm{ZFC} + \exists\kappa\ (\kappa$ *inaccessible*$))$.

Theorem

Assume AD. *Then both* ω_1 *and* ω_2 *are measurable cardinals.*

Note: Inaccessible and measurable cardinals are so large that their existence cannot be proven in ZFC alone.

Borel and Projective Hierarchies

Borel hierarchy:

- Σ^0_1: open sets
- Π^0_1: closed sets
- Σ^0_{n+1}: countable unions of Π^0_n
- Π^0_{n+1}: countable intersections of Σ^0_n

Borel hierarchy: $\bigcup_{n<\omega} \Sigma^0_n$

Projective hierarchy:

- Σ^1_1: projections of Borel sets (analytic)
- Π^1_1: complements of Σ^1_1
- Σ^1_{n+1}: projections of Π^1_n
- Π^1_{n+1}: complements of Σ^1_{n+1}

Projective hierarchy: $\bigcup_{n<\omega} \Sigma^1_n$

Determinacy and the Axiom of Choice

Borel Determinacy:

- All Borel games are determined (provable in ZF).

Projective Determinacy (PD):

- PD = every projective set is determined:

$$\mathrm{PD} := \mathrm{Det}\left(\bigcup_{n<\omega} \Pi^1_n\right)$$

- PD is not known to be provable in ZFC.
- PD does not contradict the Axiom of Choice.
- PD is consistent relative to large cardinal axioms.

Woodin Cardinals

- Woodin cardinals are extremely large cardinals, defined by how certain properties of the universe are preserved when comparing two models of set theory.
- Central to descriptive set theory and inner model theory.
- Their existence implies strong forms of determinacy.

Theorem

- $\mathrm{Con}(\mathrm{ZFC} + \mathrm{Det}(\Pi^1_2)) \iff \mathrm{Con}(\mathrm{ZFC} + 1 \text{ Woodin cardinal})$
- $\mathrm{Con}(\mathrm{ZFC} + \mathrm{PD}) \Rightarrow \mathrm{Con}(\mathrm{ZFC} + \text{n Woodin cardinals}) \quad \text{for all } n \in \omega$
- $\mathrm{Con}\left(\begin{array}{c} \mathrm{ZFC} + \text{finitely many Woodin cardinals} \\ + \text{ a measurable cardinal above them} \end{array}\right) \Rightarrow \mathrm{Con}(\mathrm{ZFC} + \mathrm{PD})$
- $\mathrm{Con}(\mathrm{ZF} + \mathrm{AD}) \iff \mathrm{Con}(\mathrm{ZFC} + \text{infinitely many Woodin cardinals})$