Picard-Kachanov type iterations for nonlinear elliptic PDEs

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1 Introduction

Last semester, we have investigated the nonlinear elliptic problem

$$\begin{cases} -\operatorname{div}(a(u)\nabla u) = f; \quad u : \Omega \to \mathbb{R}, \\ u\big|_{\partial\Omega} = 0. \end{cases}$$

This semester, our goal was to solve problems similar to that of [2]. That is, problems of the form

$$\begin{cases} -\operatorname{div}(A(x,u) \cdot \nabla u) + q(x,u)u = f; & u : \Omega \to \mathbb{R}, \\ u\big|_{\partial\Omega} = 0; & u \in \Gamma_D, \\ (A(x,u)\nabla u \cdot \nu + pu)\big|_{\partial\Omega} = g; & u \in \Gamma_N. \end{cases}$$

For unique solvability, we will have to impose the following conditions:

(i) $\Omega \subset \mathbb{R}^2$ is a bounded domain such that $\partial\Omega$ is piecewise smooth;

(ii) $A : \Omega \times \mathbb{R} \to \mathbb{R}^{2x2}$ measurable, bounded, uniformly positive;

(iii) $q : \Omega \times \mathbb{R} \to \mathbb{R}$, $p : \partial\Omega \to \mathbb{R}$ are nonnegative essentially bounded, Lipschitz continuous in the second variable;

(iv) $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$;

(v) $\partial\Omega = \Gamma_D \dot{\cup} \Gamma_N$.

As is usual, we multiply with a test function and integrate to obtain the weak formulation we are looking for $u \in H^1_D(\Omega)$:

$$\int_\Omega A(x,u) \cdot \nabla u \cdot \nabla v + \int_\Omega q(x,u)uv = \int_\Omega fv + \int_{\partial\Omega} pv \qquad \forall\, v \in H^1_D(\Omega).$$

Once again, we are solving this problem with a Picard-Kacanov-type iteration, that is, we start from some arbitrary $u_0 \in H^1(\Omega)$ and freeze the coefficients in every step. We obtain $u_{n+1} \in H^1(\Omega)$ as the solution to the (now linear) elliptic problem

$$\int_\Omega A(x,u_n) \cdot \nabla u_{n+1} \cdot \nabla v + \int_\Omega q(x,u_n)u_{n+1}v = \int_\Omega fv + \int_{\partial\Omega} pv \qquad \forall\, v \in H^1_D(\Omega),$$

with the appropriate boundary conditions. Once again, we solved these sub-problems this using the finite element method with first-order (Courant) elements. Once we determined that u_{n+1} is ”sufficiently close” (that is, the relative error of two successive iterates are closer than a threshold) we stop the iteration and accept it as the solution u of the original problem.

We need to make the following changes from the method in the previous semester:

(i) Change the construction of the stiffness matrix to account for the fact that the diffusion term is defined by a matrix-valued function instead of a functional;

(ii) Furthermore, add a term corresponding to q to the stiffness matrix;

(iii) Instead of homogeneous Dirichlet, impose the proper boundary conditions;

(iv) Change the mesh generator to account for more complicated domains.

2 The changes in detail

To make the algorithm faster, we have approached assembling the stiffness matrix triangle-wise: we can precompute the values of A and q in the nodes defining the quadrature, as well as transform the gradients from the reference element. Using these values (and the fact that the gradients are constant on every element), we can formally compute the product

$$A \cdot \nabla\varphi_j \cdot \nabla\varphi_i,$$

at every quadrature base point, and with that, we can numerically integrate the function on each triangle.

Previously, we have used a quadrature that used values in the nodes of the triangles. This was first order accurate, which was sufficient for the Courant elements. However, when evaluating the integrals

$$\int_\Omega q\varphi_j\varphi_i,$$

we can encounter second order functions, which are not handled properly by our quadrature. By interpolating values to the midpoints of the the edges, we can construct a quadrature of higher (i.e., second) order.

By adding these up and setting the respective element of the stiffness matrix to this value, we finish the construction of the stiffness matrix.

Finally, we have to account for the Robin boundary condition in the load vector. We can detect boundary edges as those who only border one triangle in the mesh. Then, since p and φ_i are analytically given, we can compute the integral

$$\int_{\partial\Omega} p\varphi_i$$

with a quadrature of arbitrarily high order.

Lastly, for the example in article [2] we had to create meshes for more generic domains. Since the domain they investigated can be composed as unifying rectangle meshes, we only had to ”stretch” our square mesh to a rectangle.

3 The experiments

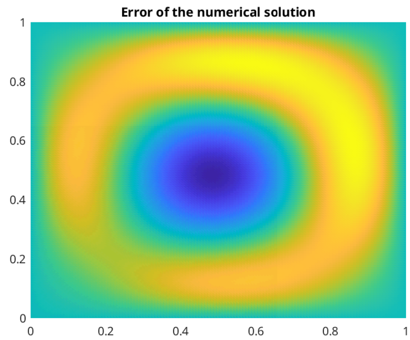
We have tried to test our method via the exact solution

$$u(x,y) = \sin(\alpha\pi x)\sin(\beta\pi y)$$

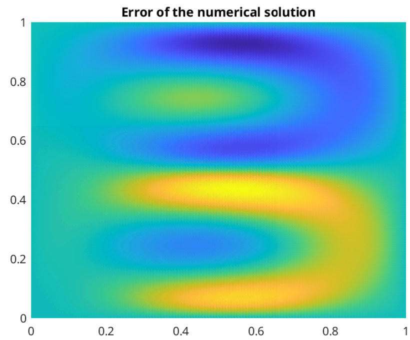
on the unit square $[0,1] \times [0,1]$, with the nonlinear functions

$$A(x,u) = \begin{pmatrix} \lambda u^2 + \|x\|_2^2 + 1 & 0 \\ 0 & u^2 + \|x\|_2^2 + 1 \end{pmatrix}, \qquad q(x,u) = 1 + \|x\|_1 + |u|$$

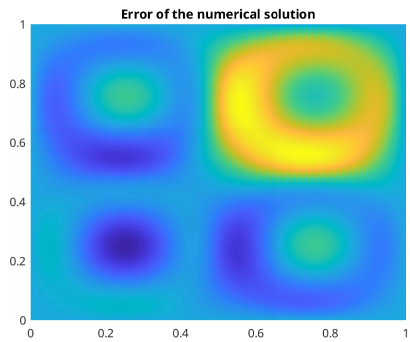
and parameters $(\alpha,\beta) \in \{(1,1),(1,2),(2,2),(2,3)\}$ (cf. 3.1).



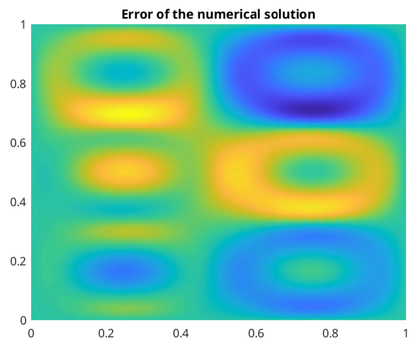
(a) $\alpha = 1$, $\beta = 1$. Maximum absolute error is 0.0778.



(b) $\alpha = 1$, $\beta = 2$. Maximum absolute error is 0.1291.



(c) $\alpha = 2$, $\beta = 2$. Maximum absolute error is 0.2172.

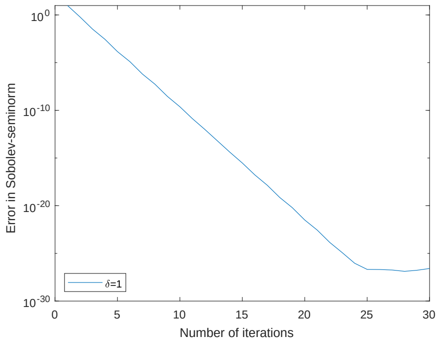


(d) $\alpha = 2$, $\beta = 3$. Maximum absolute error is 0.2179.

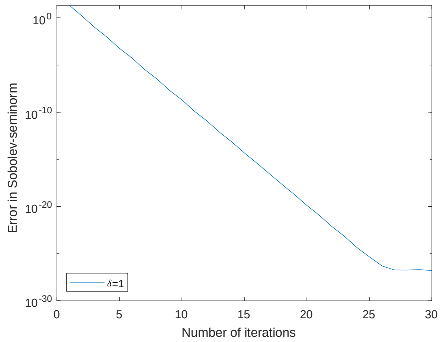
Figure 3.1: Errors of the numerical solution in max norm for different α and β values, on a 128×128 grid.

Unfortunately, we can see that we have made some mistakes, either in the construction of the stiffness matrix or calculating the analytical solution. However, we have to note that the behavior of the method is of expectance, we can observe the linear convergence in Sobolev norm in all cases (cf. 3.2).

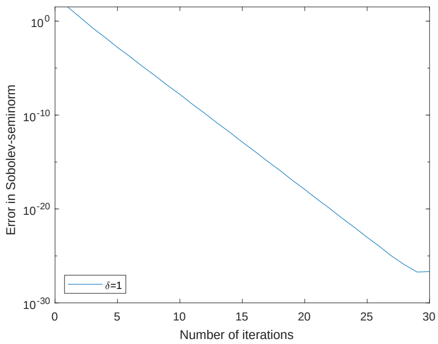
In the previous semester, we have already investigated the dependence of the diffusion term on the nonlinearity of the second variable. We can see similar behavior in this more complex case (cf. 3.3). We can quantify the speed of convergence at different δ values by estimating the slope; in order to compare with the other tests, we normalize using the first value. That is, we



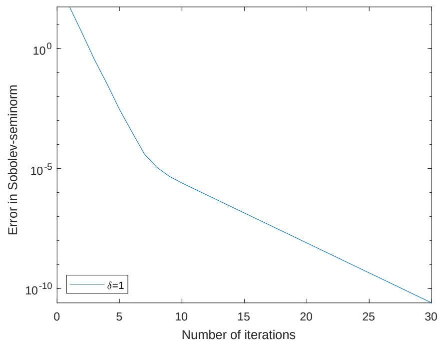
(a) $\alpha = 1$, $\beta = 1$



(b) $\alpha = 1$, $\beta = 2$



(c) $\alpha = 2$, $\beta = 2$



(d) $\alpha = 2$, $\beta = 3$

Figure 3.2: Errors in the Sobolev-seminorm at different iteration steps, on the same grid

take the slope between two points of the function on the semilog-scale, and take their average. We denote this by $m(\delta)$, and plot $\frac{m(\delta)}{m(1)}$ with respect to δ (cf. 3.4).

We will slice up the investigation from many angles. We look at how nonlinearity in the first and second variables of q (denoted by τ and ε respectively), how nonlinearity in the first variable of A (denoted by μ) and in only one of the diffusion terms (denoted by λ) affect the rate of convergence (cf. 3.5, 3.6). We do this by using

$$A(x,u) = \begin{pmatrix} \lambda\delta u^2 + \mu \|x\|_2^2 + 1 & 0 \\ 0 & \delta u^2 + \mu \|x\|_2^2 + 1 \end{pmatrix} \quad \text{and} \quad q(x,u) = \varepsilon u^2 + \tau \|x\|_2^2 + 1.$$

This ensures that all parameters are measured using similar functions, making for a better comparison.

If we plot these together (cf. 3.7), we can observe that neither μ, nor τ have any tangible effect on the rate of convergence. We can also see that the most detrimental effect is experienced in the second variable in the diffusion matrix, which is dampened a little if we only modify one of the terms. Curiously, if the reaction term is 'more nonlinear' in the second variable, the rate of convergence visibly improves.

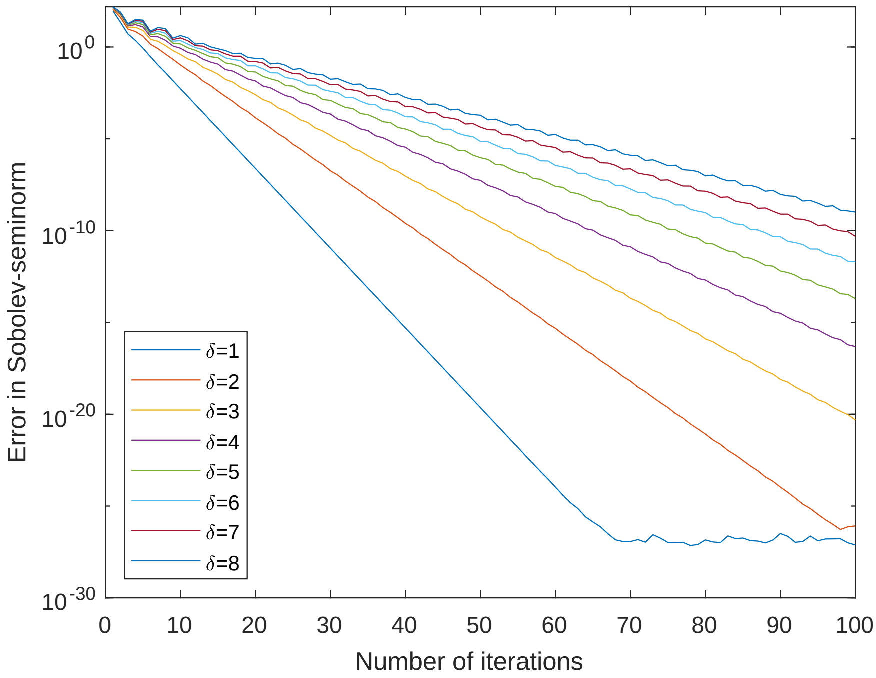


Figure 3.3: Sobolev errors at different δ values.

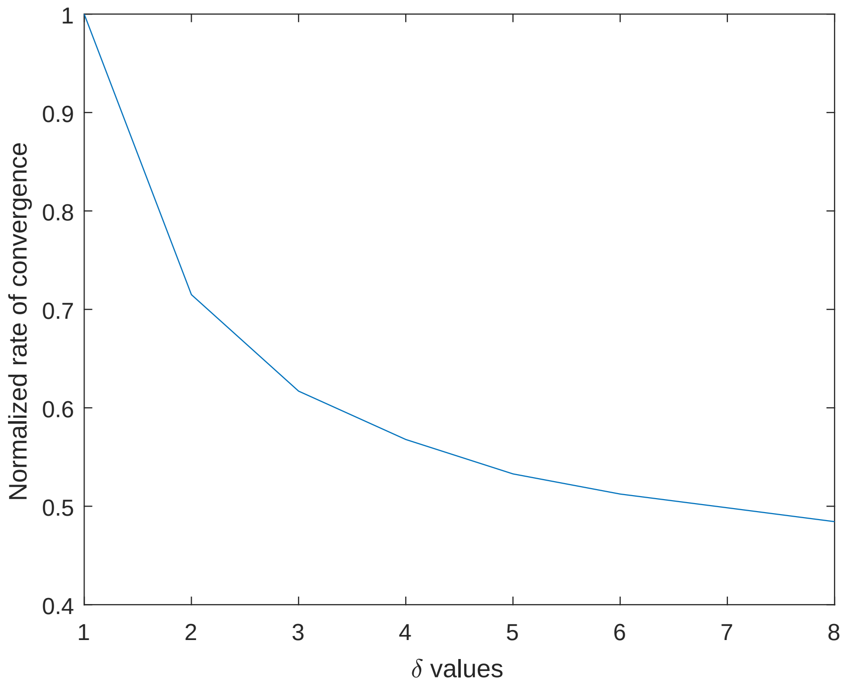
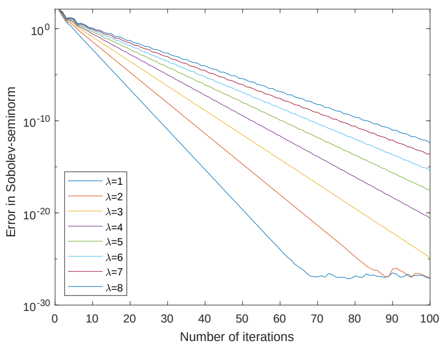
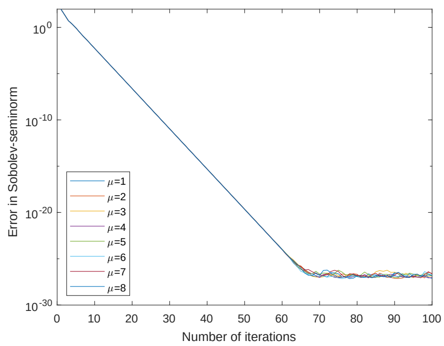


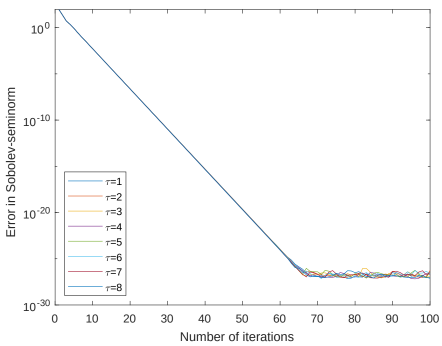
Figure 3.4: Rate of convergence at different δ values.



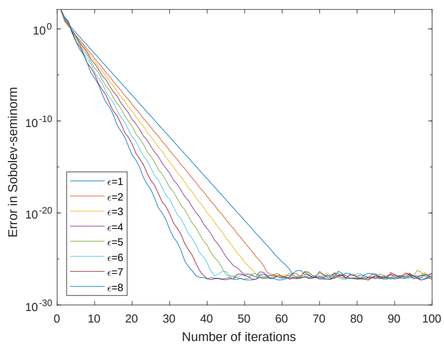
(a) λ



(b) μ

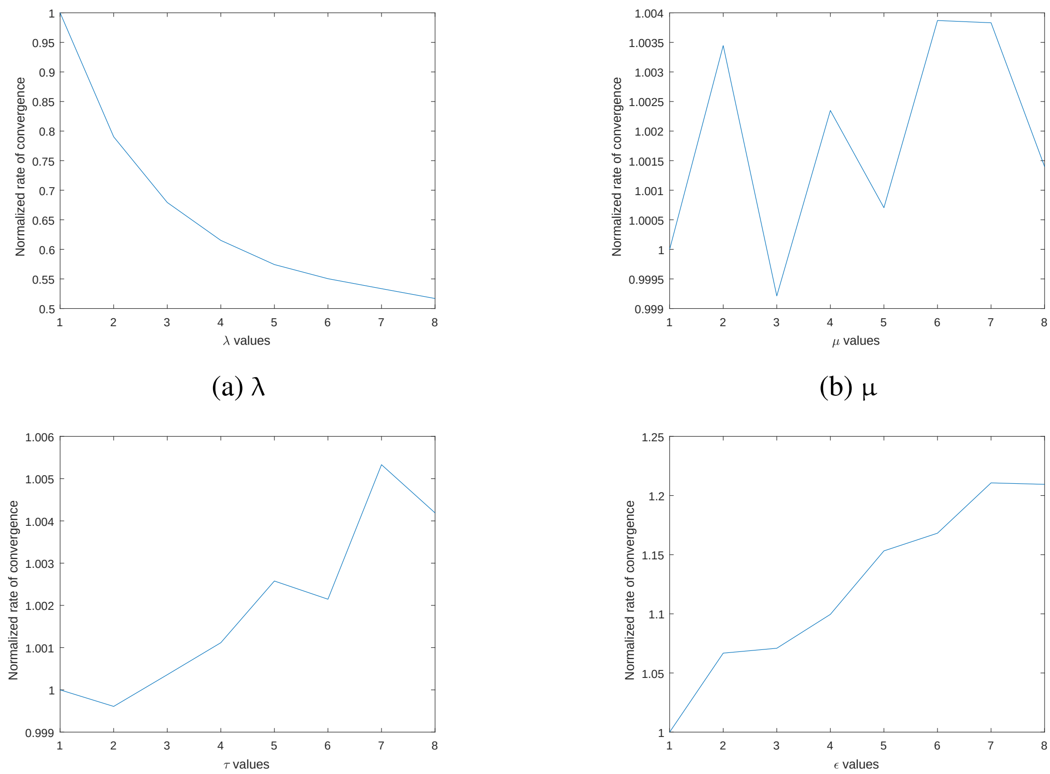


(c)]τ



(d) ε

Figure 3.5: Evolution of Sobolev errors at different kinds of nonlinearity



(a) λ

(b) μ

(c) τ

(d) ε

Figure 3.6: Rates of convergence at different kinds of nonlinearity

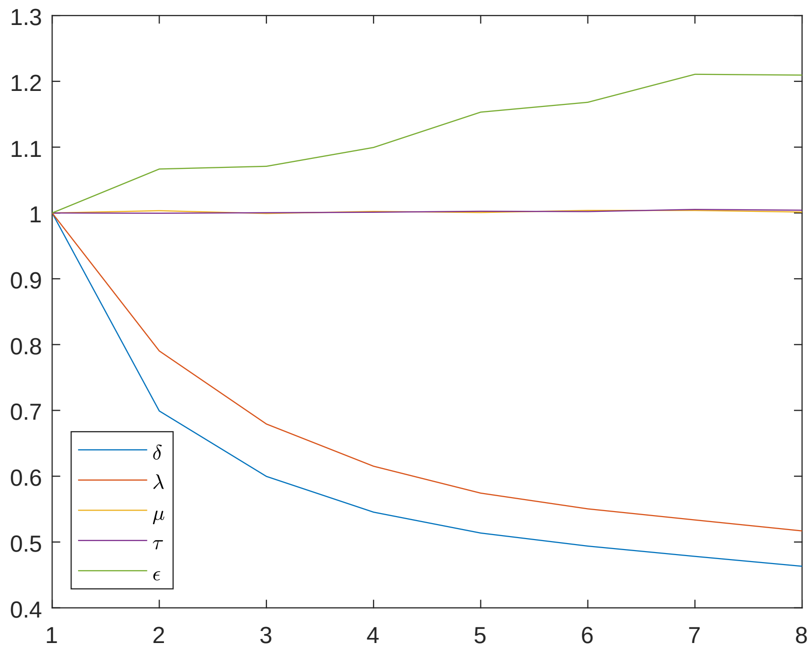


Figure 3.7: Impact of different kinds of nonlinearities

References

[1] Faragó, I., & Karátson, J. (2002). Numerical solution of nonlinear elliptic problems via preconditioning operators: Theory and applications (Vol. 11). *Nova Publishers.*

[2] Hlavácek, I., Krizek, M., & Maly, J. (1994). On Galerkin approximations of a quasilinear nonpotential elliptic problem of a nonmonotone type. *Journal of Mathematical Analysis and Applications*, 184(1), 168-189.