Picard-Kačanov-type iterations for nonlinear elliptic PDEs

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Project Work Presentation

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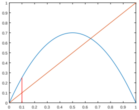


Stating the problem

Last semester, we have made looked at the equation

$$\begin{cases} -\operatorname{div}(a(u)\nabla u) = f; \quad u : \Omega \to \mathbb{R}; \\ u\big|_{\partial\Omega} = 0. \end{cases}$$

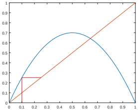
This semester, we have generalized it as follows:

$$\begin{cases} -\operatorname{div}(A(x,u)\cdot\nabla u) + q(x,u)u = f; & u : \Omega \to \mathbb{R}; \\ u\big|_{\partial\Omega} = 0; & u \in \Gamma_D; \\ (A(x,u)\nabla u \cdot \nu + pu)\big|_{\partial\Omega} = g; & u \in \Gamma_N, \end{cases}$$

which can be used to model a fuel transformer (cf. [2]).

For the FEM, we need the weak formulation

$$\int_\Omega A(x,u)\cdot\nabla u\cdot\nabla v + \int_\Omega q(x,u)uv = \int_\Omega fv + \int_{\partial\Omega} pv \qquad \forall\ v \in H^1_D(\Omega).$$



Conditions for unique solvability

The afromentioned weak problem has a unique $u \in H^1_D(\Omega)$ solution if

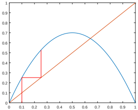
(i) $\Omega \subset \mathbb{R}^2$ is a bounded domain such that $\partial\Omega$ is piecewise smooth;

(ii) $A : \Omega \times \mathbb{R} \to \mathbb{R}^{2x2}$ measurable, bounded, uniformly positive;

(iii) $q : \Omega \times \mathbb{R} \to \mathbb{R}$, $p : \partial\Omega \to \mathbb{R}$ are nonnegative essentially bounded, Lipschitz continuous in the second variable;

(iv) $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$;

(v) $\partial\Omega = \Gamma_D \dot{\cup} \Gamma_N$.



Linearization

Instead of the nonlinear weak problem, we set some $u_0 \in H^1_D(\Omega)$, and obtain u_{n+1} by solving

$$\int_\Omega A(x,u_n)\cdot\nabla u_{n+1}\cdot\nabla v + \int_\Omega q(x,u_n)u_{n+1}v = \int_\Omega fv + \int_{\partial\Omega} pv \qquad \forall\ v \in H^1_D(\Omega).$$

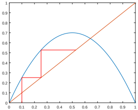
Compared to our old method, we need to account for the following:

(i) Construct the stiffness matrix such that we have matrix-valued diffusion functions;

(ii) Include a mass matrix term corresponding to q;

(iii) Impose the proper boundary conditions;

(iv) More complicated domains.

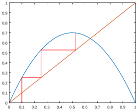


Stiffness and mass matrices

- Stiffness matrix
  - Using reference elements, we can calculate the elementwise constant gradients;
  - We can also compute $A(x, u)$ values at any base point;
  - The product $\quad A \cdot \nabla\varphi_j \cdot \nabla\varphi_i \quad$ can be used for numerical integration schemes.
- Mass matrix
  - For the reaction term, we need to add integrals of the form

$$\int_\Omega q\varphi_j\varphi_i$$

  - These can be second order polynomials;
  - Keeping the old weights and shifting the base points we can obtain a quadrature of second order.



Domain, boundary conditions

- Boundary Conditions:
  - Detect boundary edges;
  - Compute the integrals $\int_{\partial\Omega} p\varphi_i$
  - Add the values to the load vector.
- Domain:
  - We already had square domains;
  - Stretching one parameter we can obtain rectangle domains;
  - Creating unions of these cover all cases in [2].

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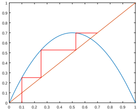
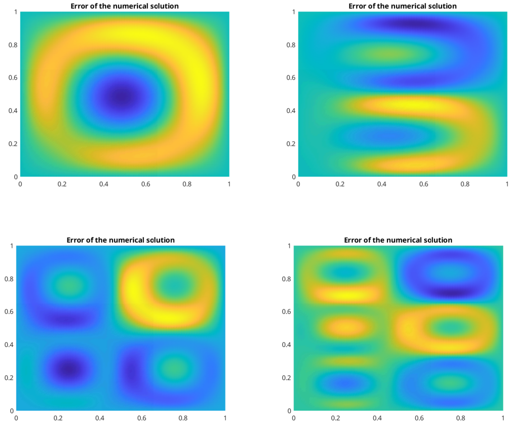
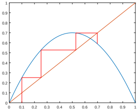
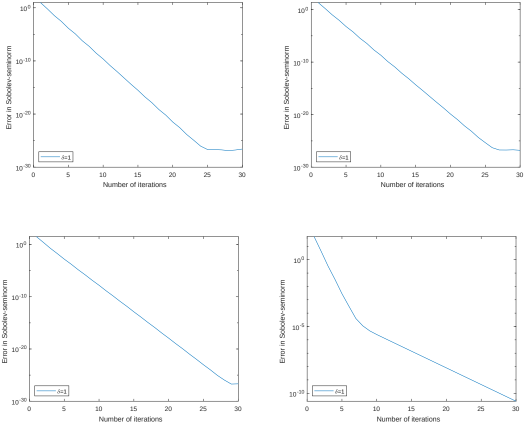


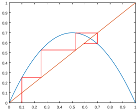
Numerical experiments, max norm





Numerical experiments, Sobolev norm





Impacts investigated

- Nonlinearity in the first variable of A – δ;
- Nonlinearity in the second variable of A – μ;
- Nonlinearity in only one diffusion term – λ;
- Nonlinearity in the first variable of q – τ;
- Nonlinearity in the second variable of q – ε.

Convergence coefficients

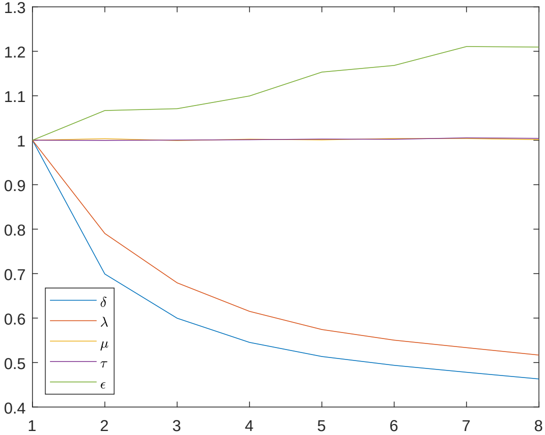


Figure: Impact of different kinds of nonlinearities

[1] Faragó, I., & Karátson, J. (2002). Numerical solution of nonlinear elliptic problems via preconditioning operators: Theory and applications (Vol. 11). *Nova Publishers.*

[2] Hlavácek, I., Krizek, M., & Maly, J. (1994). On Galerkin approximations of a quasilinear nonpotential elliptic problem of a nonmonotone type. *Journal of Mathematical Analysis and Applications*, 184(1), 168-189.