The nucleolus and related notions in cooperative games

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Introduction

During my project work, I will deal with cooperative games, which are suitable for modeling many problems like a simple market situation or a bankruptcy problem. We examine how players form a coalition in a given situation, and how the benefits obtained together could be combined later on. If a coalition has already formed, it would be worthwhile to ensure that it is not in the interest of the members or even a sub-coalition to break away from the initial one. This is kind of a stable situation, which we can describe with the concept of nucleolus, and we want to approach it as closely as possible. In my project, I will deal with this definition to find this "stable situation" and especially how it can be achieved and calculated in our model in some special cases.

Cooperative games

A cooperative game is defined by two qualities. The player set is $N = \{1, 2, \dots n\}$, where each $S \subseteq N$ subset is called a coalition. We also deal with N as the grand coalition. The other one is the characteristic function $v : 2^N \to \mathbb{R}$, which describes the value of a coalition such that $v(\emptyset) = 0$. This value is the benefits obtained by the members of the coalition. It is independent from the decisions of those not participating in the coalition. We denote a game by (N, v) in the following.

An allocation $x \in \mathbb{R}^n$ introduces a method to distribute the obtained benefit by a coalition. This assigns a value to each player but can also be defined for a coalition $x(S) = \sum_{i \in S} x_i$. An allocation x is efficient if $\sum_{i \in N} x_i = v(N)$. And x is an imputation if it is efficient and $x_i \geq v(\{i\})$ for all i. The set of efficient allocations is denoted by $I^*(N, v)$ and the imputations with $I(N, v)$. An imputation is suitable for all one-person coalition. But x can be acceptable for all coalitions: $\sum_{i \in S} x_i \geq v(S)$. If it is also efficient, we call it a core allocation. The set of core allocations is represented by the $C(N.v)$ symbol. So, in the case of a core allocation, none of the sub-coalitions is worth leaving.

We also want to differentiate between the core distributions to achieve a more stable state. For this, we define the profit $p(S, x) := x(S) - v(S)$ for each x allocation. The profit vector $\Theta(x) \in \mathbb{R}^{2^n}$ contains all the profit values of different coalitions in non-decreasing order. Profit vectors can be ordered lexicographically. The nucleolus is the only allocation that lexicographically maximizes the profit vector on $I(N, v)$:

$$N(v) = \{x \in I(N,v) | \quad \Theta(y) \preccurlyeq \Theta(x) \quad \forall y \in I(N,v)\}$$

The nucleolus always exists and is an element of the core if it is not empty. Otherwise, it is the allocation that best approximates stability. The lexicographic averaging method is an algorithm ([1]) to find the nucleolus. However, it is a slow process in general.

The bankruptcy problem

The bankruptcy problem describes a situation where a firm goes bankrupt and has to pay its creditors. Let $E \in \mathbb{R}_+$ denote the firm's liquidation value (the amount of money that can be distributed between the creditors). $c \in \mathbb{R}^n_+$ contains the claims to be paid such that $\sum_{i=1}^n c_i > E$. $x \in \mathbb{R}^n_+$ is a solution if $\sum_{i=1}^n x_i = E$. The ith agent gets x_i amount of money. Let $\mathbb{B}$ be the set of such (c, E) problems. $x(S) = \sum_{i \in S} x_i$ and $c(S) = \sum_{i \in S} c_i$ can also be defined. $r : \mathbb{B} \to \mathbb{R}^n$ is a rule and it assigns a solution to each bankruptcy problem.

This can also be formulated by using the tools of game theory. Let $N = \{1, 2, \ldots n\}$ be the set of agents, each with a claim c_i. The characteristic function can be calculated like this: $v_{(c,E)}(S) = max(E - c(N \backslash C), 0)$.

One possible rule is called the Talmud rule, which is defined as follows. The Contested Garment Principle is a distribution formula that gives a rule of a two-person game. Each agent gets what the other does not claim, and the remaining part is divided equally.

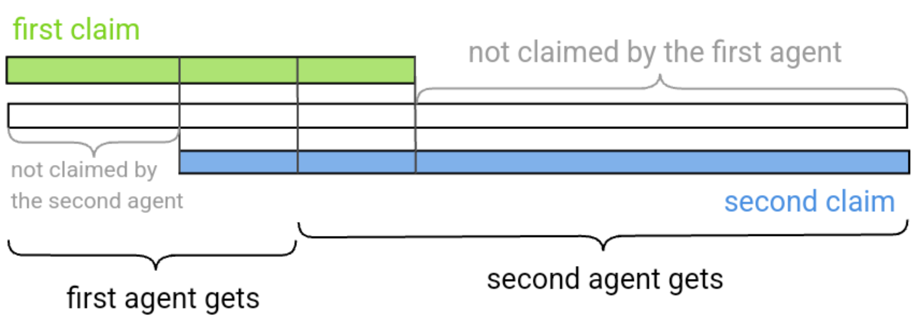

Figure 1: Contested Garment Principle

An x solution is CG-consistent if the distribution of $x_i + x_j$ for all $i \neq j$ by the Contested Garment Principle is (x_i, x_j), with the c_i, c_j claims.

CG-consistent solution (or Talmud rule):

$$T_i(c, E) = \begin{cases} min\{c_i/2, \lambda\}, & \text{if } E \leq c(N)/2 \\ max\{c_i/2, c_i - \mu\}, & \text{otherwise} \end{cases} \tag{1}$$

Hydraulic rationing

We intend to illustrate the solutions to the bankruptcy problem, which we can do with a so-called hydraulic system (intoruced in [3]). The amount of water in the system is E, which is distributed among its creditors. In the beginning, it is in a large water reservoir with a volume of E. It is connected by capillaries to n vessels, the volumes of which are the individual claims. These begin to fill up at the same time when water is poured from the tank. The average water level will be the solution to the bankruptcy problem. These vessels are based on a unit circle and have the same height. Let the height be the largest claim. Every vessel is separated into two parts by capillary except the one with the biggest claim. This way we get different distribution rules, which determine an r rule, depending on the shape of the vessels.

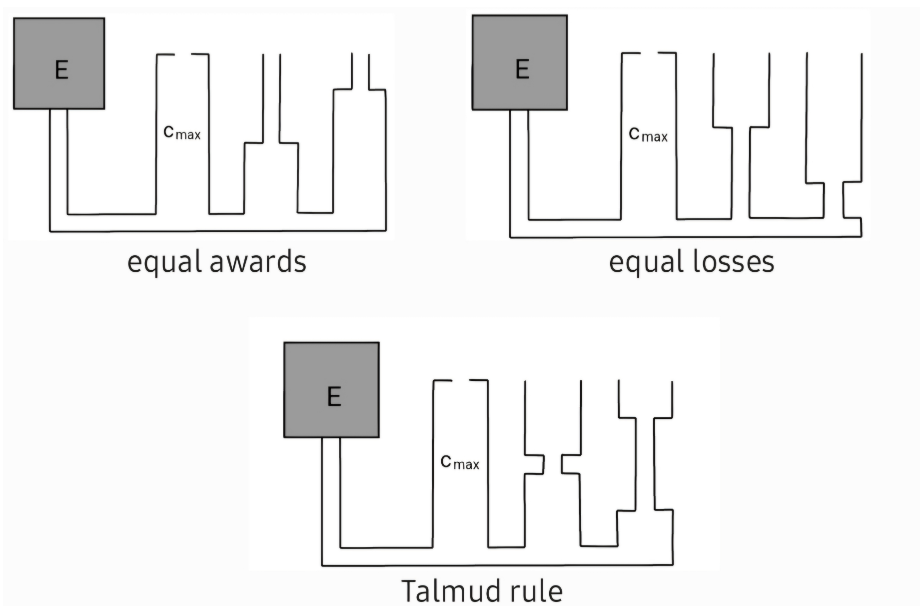

Figure 2: Representation of different rules

In the hydraulics-representing of the Talmud rule, each vessel is divided into two equal parts.

Theorem(Kaminski [5]): In the two-person bankruptcy problem, the corresponding talmudic hydraulic system's x solution is CG-consistent.The following theorem shows why it is worth dealing with the Talmud rule.

Theorem(Aumann,Mascher [4]): The CG-consistent solution to the bankruptcy problem is the nucleolus of the corresponding cooperative game.

Future research

In general, determining the nucleolus takes a lot of time, since the number of possible coalitions is exponentially large. As you could see, in the bankruptcy problem, there is an easily understandable and algorithmizable rule for finding the nucleolus. In my upcoming research, I would like to deal with other special cases where the nucleolus can be calculated in similarly simple ways.

References

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