Optimization on valuated matroids

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Modeling project work 2
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INTRODUCTION

The term matroids was first introduced in 1935 by H. Whitney. They have several equivalent definitions (using circuits, rank-functions, closed sets, etc.). The following two definitions are in terms of independent sets and in terms of the basis:

Definition 1a. A matroid is a pair $M = (S, \mathcal{F})$ consisting of a ground set S and a nonempty collection $\mathcal{F}$ of its subsets, called independent sets, satisfying the following axioms:

(1) Empty set is an independent set;

(2) Any subset of an independent set is independent;

(3) The maximal independent sets of each subset of S are equicardinal, i.e., have the same cardinality. And this maximal number is called the rank of a matroid and denoted by $r(X)$. In the paper [2] J. Edmonds reckons that from the mathematical programming point of view, the equal cardinality of all bases has special meaning — namely, that every basis is an optimum cardinality basis.

Definition 1b. A matroid is a pair $M = (S, \mathcal{B})$ consisting of a ground set S and a nonempty collection $\mathcal{B}$ of its subsets, called bases, satisfying the axioms:

(a) No proper subset of a basis is basis

(b) If B_1 and B_2 are two bases of M and $s_1 \in B_1 - B_2$. Then there exists an element $s_2 \in B_2 - B_1$ such that $B_1 - s_1 + s_2$ is a basis.

As it can be seen from the definitions, the main purpose of the matroids of that period was to clarify the notion of linear (in)dependence. It took several years that J. Edmonds [2], based on the earlier work of J. B. Kruskal and R. Rado, observed that matroids are closely related to greedy algorithms. Since then, so many interesting relations between the feasibility of greedy algorithms and different combinatorial structures have been observed and have proved to be more efficient in combinatorial optimization.

With this regard, a new notion called a valuated matroid was defined by Andreas W.M. Dress and W. Wenzel. The term valuated matroid will be abbreviated to val-matroid. Let $M = (S, \mathcal{B})$ be a matroid where $\mathcal{B}$ denotes the family of bases of M.

Recall that the exchange property of bases requires for two bases of M that, for any element $s_1 \in B_1 - B_2$, there is an element $s_2 \in B_2 - B_1$ for which $B_1 - s_1 + s_2$ is a basis of M. Actually, this is one of the basis axioms of a matroid when we define M with its bases. A useful theorem of matroids is the mutual (or symmetric) exchange property which is formulated as follows.

Proposition 1. Let B_1 and B_2 be bases of M. For any element $s_1 \in B_1 - B_2$, there is an element $s_2 \in B_2 - B_1$ for which both $B_1 - s_1 + s_2$ and $B_2 - s_2 + s_1$ are bases of M.

Lemma 2.3.13 in the book of Murota [4] states that in the basis axioms of matroids, the following weaker property implies the original basis exchange property (and hence the mutual exchange property).

For any two distinct bases B_1 and B_2, there are elements $s_1 \in B_1 - B_2$ and $s_2 \in B_2 - B_1$ for which both $B_1 - s_1 + s_2$ and $B_2 - s_2 + s_1$ are bases of M.

This is a bit nicer than the mutual exchange property since here the role of the two bases is symmetric. We may refer to this as the weak mutual exchange property.

Valuation of matroid.

Let ω be a function on 2^S which is finite on the basis $\mathcal{B}$, and is $-\infty$ on every subset X of S which is not a basis. The function ω is called a **valuation** of M (or the bases of M), if it is discrete concave in the sense that for any two bases B_1, B_2 and for any element $s_1 \in B_1 - B_2$, there exists an element $s_2 \in B_2 - B_1$ for which

$$\omega(B_1) + \omega(B_2) \leq \omega(B_1 - s_1 + s_2) + \omega(B_2 - s_2 + s_1). \tag{1}$$

Let us consider some examples of the valuation:

Example 1. We are given the matrix matroid when the terms of the matrix are polynomial in variables x. We consider the columns and rows of the matrix as linearly independent sets. We take the maximum number of linearly independent columns. Here column set is a ground set. If the matrix has m rows and they are linearly independent, the the rank of this matrix is equal to m. Then we take m linearly independent columns. Determinant of a matrix is not equal to 0. The largest degree of the determinant(which is, in turn, polynomial in variable x belonging to the basis) is the valuation of the given matroid.

Let us consider the following matrix:

$$P = \begin{pmatrix} x & x^2 \\ 1+x & x^3 \end{pmatrix}$$

$det(P) = x^4 - x^2 - x^3$. The largest degree of this polynomial is 4. Therefore, the valuation of this basis is 4.

Example 2. Here we are given an edge-weighted bipartite graph $G = (S, T; E)$. This defines a matroid M on S (this is the so-called transversal matroid) in which a subset $I \subseteq S$ is independent by definition if there is a matching of G covering I. The valuation of a basis $B \subseteq S$, by definition is the maximum weight matching covering B. Then this is a valuation satisfying the axioms.

We say that a basis B is an **ω-maximizer** if $\omega(B) \geq \omega(B')$ holds for every basis B' of M.

Lemma 1. A basis B is an ω-maximizer if and only if

$$\omega(B) \geq \omega(B - s + t) \text{ holds for every } s \in B, t \in S - B. \tag{2}$$

Proof. If $\omega(B) < \omega(B-s+t)$ holds, then $\omega(B-s+t)$ is finite, that is, $B' := B-s+t$ is a basis of M, showing that B is not a ω-maximizer.

Conversely, assume that (2) holds. Suppose indirectly that B is not an ω-maximizer. Let B' be an ω-maximizer basis for which $|B' \cap B|$ is as large as possible. Let $s \in B - B'$ be an element. By (1), there is an element $s' \in B' - B$ for which $\omega(B) + \omega(B') \leq \omega(B - s + s') + \omega(B' - s' + s)$. Since B' is an ω-maximizer, we have $\omega(B' - s' + s) \leq \omega(B')$. Here we cannot have equality, since then $B'' := B' - s' + s$ would also be an ω-maximizer for which $|B'' \cap B| = |B' \cap B| + 1$, contradicting the choice of B'. Therefore $\omega(B' - s' + s) < \omega(B')$, and hence

$$\omega(B) + \omega(B') \leq \omega(B - s + s') + \omega(B' - s' + s) < \omega(B - s + s') + \omega(B'),$$

from which $\omega(B) < \omega(B - s + s')$, contradicting the hypothesis of the lemma. ∎

If $\omega(B) < \omega(B - s + t)$ holds for a pair s, t of elements with $s \in B, t \in S - B$, then the operation of replacing basis B with basis $B' := B - s + t$ is called a **local improvement**.

Lemma 1 implies that if we start with an arbitrary basis of M and apply local improvements as long as possible, then the final basis is an ω-maximizer. In other words, this algorithm considers at a general step the current basis B and checks by considering all the $|B||S - B|$ pairs $s \in B, t \in S - B$ of elements, if there is such a pair with $\omega(B) < \omega(B - s + t)$. If there is one, then the algorithm continues with basis $B' := B - s + t$ in place of B. If no such a pair $\{s, t\}$ exists, then Lemma 1 ensures that B is an ω-maximizer. We refer to this simple as the **local improvement algorithm**.

If K denotes the number of distinct ω-values, then the total number of pairs $\{s, t\}$ considered for possible local change during the algorithm is $O(Kr(n - r))$, where r is the rank of M and $n = |S|$. This implies that the local improvement algorithm is polynomial if K is small (in the sense that K can be bounded by a polynomial of n), but nothing is known in the general case.

There is intuitive idea to speed up the local improvement algorithm. Namely at each step, choose the pair $\{s, t\}$ in such a way that the increment $\omega(B - s + t) - \omega(B)$ is as large as possible. This approach is termed in the literature as the steepest ascent method.

Lemma 2. (Shioura.) Let B be a basis of M and $Z \subset B$ a (possible empty) subset of B for which there is an ω-maximizer basis including Z. Let s be an element of $B - Z$. (A) If $\omega(B) \geq \omega(B - s + t)$ for each $t \in S - B$, then there is an ω-maximizer basis including $Z + s$. (B) If there is an element $t \in S - B$ for which $\omega(B) < \omega(B - s + t)$ and t is chosen in such a way that $\omega(B - s + t)$ is as large as possible, then there is an ω-maximizer basis including $Z + t$.

Proof. (A) Let $B_{\max}$ be an ω-maximizer basis including Z for which $|B_{\max} \cap B|$ is as large as possible.

Claim 1. $s \in B_{\max}$.

Proof. Suppose indirectly that $s \notin B_{\max}$, that is, $s \in B - B_{\max}$. By (1), there is an element $t \in B_{\max} - B$ for which

$$\omega(B) + \omega(B_{\max}) \leq \omega(B - s + t) + \omega(B_{\max} - t + s). \tag{3}$$

Since $B_{\max}$ is an ω-maximizer, $\omega(B_{\max}) \geq \omega(B_{\max} - t + s)$. But here we cannot have equality, since then $B'_{\max} := B_{\max} - t + s$ is also an ω-maximizer for which $|B'_{\max} \cap B| = |B_{\max} \cap B| + 1$, contradicting the choice of $B_{\max}$.

Therefore $\omega(B_{\max}) > \omega(B_{\max} - t + s)$. But this and (4) imply $\omega(B) < \omega(B - s + t)$, contradicting the hypothesis of Case (A). ■

(B) Let $B' := B - s + t$. and $B_{\max}$ be an ω-maximizer basis including Z for which $|B_{\max} \cap B'|$ is as large as possible.

Claim 2. $t \in B_{\max}$.

Proof. Suppose indirectly that $t \notin B_{\max}$, that is, $t \in B' - B_{\max}$. By applying (1) to B' and $B_{\max}$ with t in place s and with v in place t, we obtain that there is an element $v \in B_{\max} - B'$ for which

$$\omega(B') + \omega(B_{\max}) \le \omega(B' - t + v) + \omega(B_{\max} - v + t). \tag{4}$$

Since $B_{\max}$ is an ω-maximizer, $\omega(B_{\max}) \ge \omega(B_{\max} - t + s)$. But here we cannot have equality, since then $B'_{\max} := B_{\max} - v + t$ is also an ω-maximizer for which $|B'_{\max} \cap B| = |B_{\max} \cap B| + 1$, contradicting the choice of $B_{\max}$.

Therefore $\omega(B_{\max}) > \omega(B_{\max} - v + t)$. But this and (4) imply $\omega(B - s + t) = \omega(B') < \omega(B - s + v)$, contradicting the choice of t. ■■

This lemma was proved by Shioura in the special case when $Z = 0$. In the lemma, the two cases can be discussed in a more concise, unified way (as was done by Shioura) but the separation of the two cases makes easier to understand the proof.

Theorem 1. The local improvement algorithm, when the steepest ascent rule is applied for selecting the subsequent basis $B - s + t$, is strongly polynomial.

A proof of this result can be found in [5]. For an intermediate basis B, the algorithm must check all pairs $\{s, t\}$ with $s \in B, t \in S_B$ to select the one maximizing the increase of ω. This requires $|r||n - r|$ evaluations of ω, and hence the total number of ω-evaluations is $r^2(n - r) + 1$.

Dress-Wenzel algorithm.

The following algorithm, due to Dress and Wenzel uses a slightly more complicated rule in the local improving algorithm for selecting the pair $\{s, t\}$, but it has the advantage that it requires altogether at most $r(n - r) + 1$ ω-evaluations of ω.

The greedy algorithm of Dress and Wenzel for val-matroids runs as follows. We start with a basis $B = \{s_1, s_2, \ldots, s_r\}$ and use the given order of its elements. The algorithm consists of r phases. In Phase i we consider element s_i and decide whether s_i should remain in the current basis or it is changed by an element t_i from outside of the current basis. If s_i remains in the basis, then this is a final decision, if s_i is replaced by t_i then it is final that t_i remains in the basis (but the element s_i falling out from the basis may return at phase j $(j > i)$ as an element t_j. An example is shown below to demonstrate this returning phenomenon.)

To be more specific, let $B_0 := B$ and consider Phase 1. Check whether there is an element t in $S - B_0$ for which $\omega(B_0) < \omega(B_0 - s_1 + t_1)$. If no such an element exists, then we define $B_1 := B_0$ (that is, we keep s_1 in the current basis), and turn to Phase 2. If there is such a t, then we choose one, denoted by t_1, for which $\omega(B_0 - s_1 + t_1)$ is as large as possible and replace s_1 by t_1, that is, $B_1 := B_0 - s_1 + t_1$.

To describe Phase k $(k \ge 2)$, suppose that basis B_{k-1} has been constructed in the previous Phase $k - 1$. (Note that $\{s_k, s_{k+1}, \ldots, s_r\} \subseteq B_{k-1}$). Check whether there is an element t in $S - B_{k-1}$ for which $\omega(B_{k-1}) < \omega(B_{k-1} - s_k + t)$. If no such an element exists, then we define $B_k := B_{k-1}$ (that is, we keep s_k in the current basis). If $k = r$, the algorithm terminates by outputting B_r, while if $k < r$, we turn to Phase $k + 1$. If there is such an element t, then we choose one, denoted by t_k, for which $\omega(B_{k-1} - s_k + t_k)$ is as large as possible, and replace s_k by t_k, that is, $B_k := B_{k-1} - s_k + t_k$.

Theorem 2. The final basis B_r output by the algorithm is an ω-maximizer.

Proof. This is an immediate consequence of Shioura Lemma. Consider the basis B_1 resulted by Phase 1 and let z_1 denote the first element B_1. Then either $z_1 = s_1$ or $z_1 = t_1$ (where $t_1 \in S - B_0$ is the element selected by Phase 1 to replace s_1). By Lemma 2, there exists an ω-maximizer basis containing z_1.

For the general case, consider Phase k. Let $Z := \{z_1, \ldots, z_{k-1}\}$ be the set of the first $k-1$ elements of the current basis B_{k-1} determined by Phase $k-1$. By induction, we can assume that Z is a subset of an ω-maximizer basis. Let z_k denote the k'th element of B_k. Then either $z_k = s_k$ or $z_k = t_k$ (where $t_k \in S - B_{k-1}$ is the element selected by Phase k to replace s_k). By Lemma 2, there exists an ω-maximizer basis including Z.

At termination in Phase r, the final Z is B_r, and since Z is included by ω-maximizer basis, the basis B_r is an ω-maximizer. ∎

Let us now show an example of how Dress-Wenzel algorithm works.

Example 3. Suppose that we have a graphic matroid (complete graph K_4) with edges denoted by 1,2,3,4,5,6. Consider the linear weight where the weight of edge i is i. In this

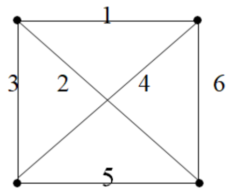

Figure 1
K_4 matroid

case the unique max-weight basis (spanning tree) is $\{6, 5, 3\}$.

Suppose that the starting basis of the algorithm is $\{3, 2, 1\}$ in this order. Then the greedy algorithm of Dress-Wenzel at the very first step puts edge 5 to the place of edge 3. That is, at this moment edge 3 falls out of the basis. But later edge 3 must come back since the unique max wight basis is $\{6, 5, 3\}$.

This basically means that an element removed at a certain moment from the current basis may come back later. It is true, however, that if an element t has been added at a certain moment to the basis, then it will stay in the basis forever. This example demonstrates an assymmetry between the leaving and coming elements: an element that leaves the current basis at a certain moment may come back, while an element entering the basis at a certain moment will never leave the basis later.

Bibliography

[1] A.W.M. Dress and W. Wenzel, *Valuated Matroids*, Advances in Mathematics, 93, (1992) 214-250

[2] J. Edmonds, *Matroids and the greedy algorithm*, Math. Programming, 1 (1971) 127-136.

[3] A. Frank, Connections in Combinatorial Optimization, Oxford University Press, 2011 (ISBN 978-0-19-920527-1). Oxford Lecture Series in Mathematics and its Applications, 38.

[4] K. Murota, Matrices and matroids for system analysis, Springer (2000), Ser. Algorithms and Combinatorics, 20.

[5] K. Murota, Discrete Convex Analysis, SIAM, Philadelphia, 2003.

[6] R. Rado, *A theorem on independence relations,* The Quarterly Journal of Mathematics (Oxford) (2) 13 (1942) 83-89.

[7] A. Shioura, *Minimization of an M-convex function*, Discrete Applied Mathematics, 84 (1998) 215-220.

[8] A. Schrijver, Combinatorial Optimization: Polyhedra and Efficiency, Springer, 2003. Vol 24. of the series Algorithms and Combinatorics.