Picard-Kachanov type iterations for nonlinear elliptic PDEs

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1 Introduction

In this semester we have investigated an iterative method for the nonlinear elliptic partial differential equation

$$\begin{cases} -\mathrm{div}(a(u)\nabla u) = f; \quad u : \Omega \to \mathbb{R}, \\ u\big|_{\partial\Omega} = 0, \end{cases}$$

where

(i) $\Omega \subset \mathbb{R}^2$ is a bounded domain such that $\partial\Omega$ is piecewise smooth;

(ii) $a : \mathbb{R} \to \mathbb{R}$ is measurable, bounded, uniformly positive;

(iii) $f \in L^2(\Omega)$.

We first state the weak formulation of the problem: we are looking for $u \in H_0^1(\Omega)$ such that

$$\int_\Omega a(u)\nabla u \cdot \nabla v = \int_\Omega fv \qquad \forall\, v \in H_0^1(\Omega).$$

Since the argument of a is the unknown function, we will use an iterative method to approximate u, that is, given some starting value u_0, we get u_{n+1} for all n as the solution of the linearized problem

$$\int_\Omega a(u_n)\nabla u_{n+1} \cdot \nabla v = \int_\Omega fv \qquad \forall\, v \in H_0^1(\Omega),$$

where all u_n are members of $H_0^1(\Omega)$.

2 The Numerical Method

The numerical solution of the above problem is most often obtained using the finite element method, that is, we solve the integral equation in some subspace $V_h \subset H_0^1(\Omega)$, where h corresponds to the mesh width. For the sake of brevity, we will only consider the Courant subspace and only touch on the differences caused by the extra *frozen coefficient* term.

In this simple case, the difference only manifests in the construction of the stiffness matrix. For each basis pair φ_j, φ_i we have to calculate (approximate) the bilinear form

$$\int_\Omega a(u_n)\nabla\varphi_j \cdot \nabla\varphi_i.$$

Using the reference element method, we get that

$$\begin{aligned}\int_T a(u_n)\nabla\varphi_j \cdot \nabla\varphi_i &= |\det(Z)| \int_R a(u_n \circ \xi)\,(\nabla\varphi_j)\circ\xi \cdot (\nabla\varphi_i)\circ\xi \\ &= |\det(Z)|\, Z^{-1}\nabla\widehat{\varphi}_j \cdot Z^{-1}\nabla\widehat{\varphi}_i \int_R a(u_n \circ \xi)\end{aligned}$$

and since the value of $Z^{-1}\nabla\widehat{\varphi}_j \cdot Z^{-1}\nabla\widehat{\varphi}_i$ is a known constant on the reference triangle, we can factor out and we only have to evaluate

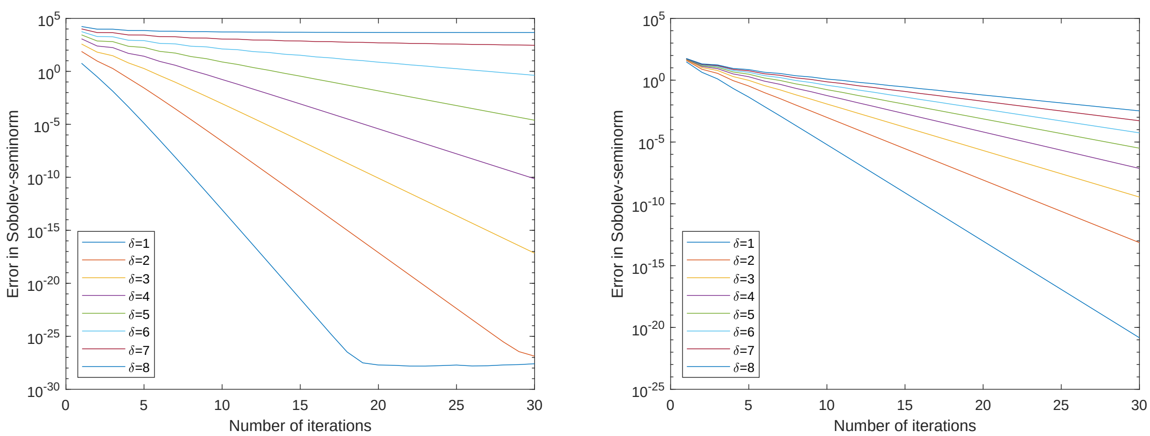
$$\int_R a(u_n \circ \xi),$$

which we have chosen to do with a second-order three-point quadrature using the values at the nodes. (Higher order quadratures are possible, however, with Courant elements this does not improve the accuracy. Moreover, our scheme is practical since in the FEM subspace the values of u_n are already known in the nodes.)

Lastly, we constructed the iteration as follows: we set $u_0 \in V_h$ to be the solution of the problem with $a = 1$ (i.e. we solve for a Poisson equation), and we use the above method to calculate $u_{n+1} \in V_h$. Once the relative maximum norm of $u_n - u_{n+1}$ is smaller than a given threshold or we have iterated for too long *(number of iterations, time)*, we accept the last u_{n+1} as the solution.

3 The Experiments

We have conducted tests on two problems, both with the nonlinear function $a(u) = \delta u^2 + 1$ with sinusoidal *(left)* and Gaussian *(right)* right hand sides. We can observe that the method



reproduces the expected linear convergence and that the quotient of convergence is worse (larger) as δ grows, i.e. when the problem is farther from linear.

References

[1] Faragó, I., & Karátson, J. (2002). Numerical solution of nonlinear elliptic problems via preconditioning operators: Theory and applications (Vol. 11). *Nova Publishers.*

[2] Hlavácek, I., Krizek, M., & Maly, J. (1994). On Galerkin approximations of a quasilinear nonpotential elliptic problem of a nonmonotone type. *Journal of Mathematical Analysis and Applications*, 184(1), 168-189.