Picard-Kačanov-type iterations for nonlinear elliptic PDEs

Sike András

Project Work Presentation

Eötvös Loránd Tudományegyetem

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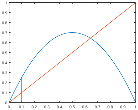


Stating the problem

We have been investigating the equation

$$\begin{cases} -\mathrm{div}(a(u)\nabla u) = f; \quad u : \Omega \to \mathbb{R}, \\ u\big|_{\partial\Omega} = 0, \end{cases}$$

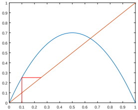
or in weak form,

$$\int_\Omega a(u)\nabla u \cdot \nabla v = \int_\Omega fv \qquad \forall\ v \in H_0^1(\Omega).$$

We treat this nonlinear equation with a numerical iteration

$$\int_\Omega a(u_n)\nabla u_{n+1} \cdot \nabla v = \int_\Omega fv \qquad \forall\ v \in H_0^1(\Omega),$$

which converges linearly, globally.



The numerical framework

Solving the above equation is done using regular FEM techniques. However, we have to be careful when assembling the stiffness matrix:

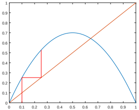
$$\int_\Omega a(u_n)\nabla\varphi_j \cdot \nabla\varphi_i$$

elements can be formally calculated as

$$\begin{aligned}\int_T a(u_n)\nabla\varphi_j \cdot \nabla\varphi_i &= |\det(Z)| \int_R a(u_n \circ \xi)\,(\nabla\varphi_j)\circ\xi \cdot (\nabla\varphi_i)\circ\xi \\ &= |\det(Z)|\, Z^{-1}\nabla\widehat{\varphi}_j \cdot Z^{-1}\nabla\widehat{\varphi}_i \int_R a(u_n \circ \xi),\end{aligned}$$

so only this expression has to be evaluated:

$$\int_R a(u_n \circ \xi).$$



The test problems

We have conducted experiments on two problems, both with the nonlinearizing function

$$a(u) = \delta u^2 + 1.$$

We measured the ratio of successive errors in the Sobolev-norm and got the following results:

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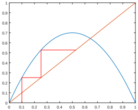
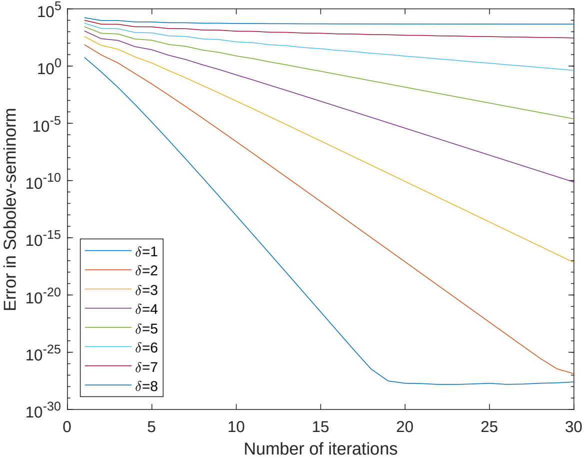


First Comparison



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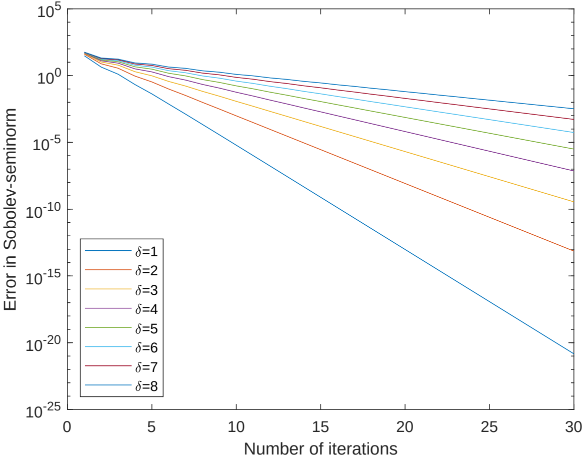
Second comparison



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[1] Faragó, I., & Karátson, J. (2002). Numerical solution of nonlinear elliptic problems via preconditioning operators: Theory and applications (Vol. 11). *Nova Publishers.*

[2] Hlavácek, I., Krizek, M., & Maly, J. (1994). On Galerkin approximations of a quasilinear nonpotential elliptic problem of a nonmonotone type. *Journal of Mathematical Analysis and Applications*, 184(1), 168-189.