Graphs with integer edges on the plane

Gyenizse-Nagy András (E4FJ13)

December 15, 2024

1 Overwiew

In this semester I studied graphs drawn in the plane with edges of integer length, specifically complete graphs. In other words, these are point sets $P = \{P_1, P_2, \ldots, P_n\}$ in the plane, where $dist(P_i, P_j)$ is an integer for every $i, j \leq n$. For simplicity we will call them *integral point sets.*

Remark. Notice that the problem of finding finite integral point sets and finding finite rational point sets, where every distance is a rational number is the same. For a rational point set we can enlarge the distances with the common denominator, thus we get an integral point set.

There are many open questions and problems in this topic, the main one is from Erdős, which is about finding integral point sets with the most number of points where there are no 3 points on a line and no 4 points on a circle. For now the biggest known sets contains 7 points. Another famous open problem is from Erdős and Ulam, that asks for a dense set in the plane such that all pairwise distances are rational [5]. Also, a connected problem is formulated in the Harborth conjecture, which states that every planar graph has a planar drawing in which every edge is a straight segment of integer length.

Although there are many unsolved problems, a few steps have been taken. There are constructions of point sets with arbitrary many points on a line or on a circle that have a pairwise rational distance. Erdős and Anning proved that whenever an infinite number of points in the plane all have integer distances, the points lie on a straight line [1].

One of the approach to find a large integral point set is to generating a list of sets with 3 points, and comparing them in order to find 2-2 points with the same distances. Then checking the remaining 2 points whether they have integer distance, and in that case we can combine them into a 4 point set (see Figure 1). Then repeat the same method with larger and larger sets. It is a slow, exhausting method and requires a lot of computational power, but the first integer point sets containing 7 points were found this way [3].

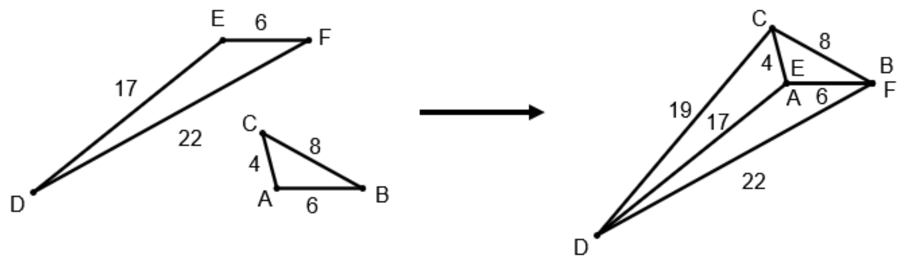


Figure 1: generating a four point set

If we compute the smallest possible diameter for n points, where the diameter is the largest occurring distance in a point set, we get 1, 8, 73, 174, 2262000 for $n = 3, 4, 5, 6, 7$ respectively [2][3]. We can see that the size is rapidly increasing, which makes the search progressively harder.

In this document we will use the *characteristic* of a triangle, that will be useful later to find integral point sets. We show that in an integral point set, all triangle has the same characteristics. We also discuss parametrized point sets, where the distances are given as a polynomial of parameters. This way we can create infinitely many integer point sets using different parameters, so it is a very useful approach.

2 Characteristic of a triangle

In this section we introduce some simple but useful lemmas and concepts.

Using Heron's formula, given the sides of a triangle, we can easily compute the area. If the three sides of a triangle are all integer, we can write up the area in a $\frac{1}{4}\sqrt{p_1^{k_1} p_2^{k_2} \dots p_n^{k_n}}$ form, where $p_1, p_2, \dots, p_n$ are different primes. We can transform this into the following form:

$$\frac{1}{4}L\sqrt{\prod p_i}, \quad L \in \mathbb{Z},\ p_i \text{ are those primes, where } k_i \text{ was odd.}$$

Definition 2.1. If the area of an integer sided triangle is $A_\triangle = l\sqrt{k}$, where $l \in \mathbb{Q}$ and k is a square-free integer, then the *characteristic* of a triagle is k. Later we denote the characteristic of the ABC triangle with K_{ABC}.

Notice, that the formula above is unique for a given triangle, thus K_{ABC} is well defined if the triangle is not degenerate.

Lemma 2.2. In an integer-sided triangle the cosine of every angle is rational.

Proof. We use the cosine theorem:

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

$$\frac{a^2 + b^2 - c^2}{2ab} = \cos\gamma$$

Since $a,\ b,\ c \in \mathbb{Z}$, the left side is rational, thus also the right side. □

Theorem 2.3. *In an integral point set, any two triangles (determined by 3-3 points) have the same characteristic.*

Proof. It is enough to show this for two triangles that share two vertices, because then if two triangles are $A_1A_2A_3$ and $B_1B_2B_3$, where every point is different, then $K_{A_1A_2A_3} = K_{A_1A_2B_3} = K_{A_1B_2B_3} = K_{B_1B_2B_3}$.

Let the two triangles be ABC and ABD, with side lengths a_1, b_1, c and a_2, b_2, c and with directed angles $\angle CAB = \alpha_1$ and $\angle BAD = \alpha_2$ in the common A vertex, and let α denote the directed angle $\angle CAD$ (see Figure 2).

Then $\alpha = \alpha_1 + \alpha_2$. The triangles ABC, ABD and ACD have areas respectively $\frac{1}{2}b_1c\,|\sin\alpha_1|$, $\frac{1}{2}b_2c\,|\sin\alpha_2|$ and $\frac{1}{2}b_1b_2\,|\sin\alpha|$, (the absolute value is due to the directed angle), and we can write them up in the form $l\sqrt{k}$, where $l \in \mathbb{Q}$ and k is a square-free integer. This implies that we can determine the characteristic from the sine values. Let $\sin\alpha_1 = l_1\sqrt{k_1}$, $\sin\alpha_2 = l_2\sqrt{k_2}$, $\sin\alpha = l_3\sqrt{k_3}$, with $l_1, l_2, l_3 \in \mathbb{Q}$ (not necessarily positive now) and k_1, k_2, k_3 are square-free integers.

$$l_3\sqrt{k_3} = \sin\alpha = \sin(\alpha_1+\alpha_2) = \sin\alpha_1\cos\alpha_2 + \sin\alpha_2\cos\alpha_1 = l_1\sqrt{k_1}\cos\alpha_1 + l_2\sqrt{k_2}\cos\alpha_2$$

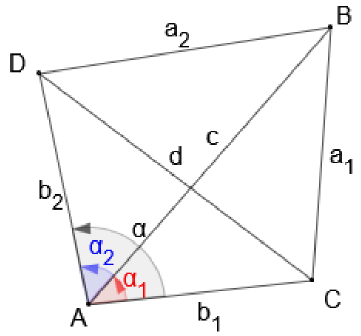
$$l_3^2k_3 = l_1^2(\cos\alpha_1)^2k_1 + l_2^2(\cos\alpha_2)^2k_2 + 2l_1l_2\cos\alpha_1\cos\alpha_2\sqrt{k_1k_2}$$

$$\frac{l_3^2k_3 - l_1^2(\cos\alpha_1)^2k_1 - l_2^2(\cos\alpha_2)^2k_2}{2l_1l_2\cos\alpha_1\cos\alpha_2} = \sqrt{k_1k_2}$$

By Lemma 2.2, the cosines are rationals, so the left side is rational. Because $k_1k_2 \in \mathbb{Z}_+$, and the square root of a positive integer is either integer or irrational, $\sqrt{k_1k_2}$ must be integer, thus k_1k_2 a square. Since k_1 and k_2 are square free, they must be equal. They were the characteristics of the triangles ABC and ABD, so the claim is proven. □

3 Parametrization of the distances

Now we are interested in the case when the distances in the integral point set are given in a parametrized form, specifically the sides are polynomials of the parameters, so $dist(P_i, P_j) \in \mathbb{Z}[x]$. This is not a new idea; parametric solutions are known for Pythegorian triples and Heronian triangles. The first integer point sets with seven points in general position were found using exhaustive search [3], but that resulted in only finitely many constructions. Later Trinh Xuan Minh created a parametrized construction, which yields a valied construction for infinitely many values of the parameters. [4]. This motivated us to study parametrized constructions. The idea of creating bigger constructions by combining smaller ones also works here.



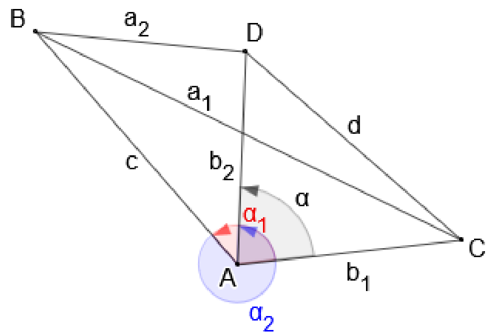


Figure 2: two different placings of the four point

In this semester I generalized the lemmas in section 2 for the parametric integer point sets. There can be different interpretations, here we only work with one variant polynomials. Using Heron's formula, now the characteristic will be a $K(x) \in \mathbb{Z}[x]$ polynomial, which is a product of monic, square free irreducible polynomials for uniqueness.

Theorem 3.1. *In an integral point set with polynomial parametrization, every two triangle (determined by 3-3 points) have the same $K(x) \in \mathbb{Z}[x]$ characteristic.*

Proof. Similar to Theorem 2.3, but here every distance is a polynomial, instead of the rational numbers there are fractions of polynomials, under the squares are product of monic, square free irreducible polynomials. □

Theorem 3.2. *Every polynomial integral point set can be written in a coordinate system, where all point has coordinates in the form $(p(x),\ q(x)\sqrt{k(x)})$, where $p(x),\ q(x),\ k(x) \in \mathbb{Z}[x]$, $k(x)$ is monic and square free. Also, $k(x)$ is the same for every point.*

Proof. If there are 1 or 2 points, the theorem is trivial. For more points, we fix two of them, A and B with distance $c(x)$ at the coordinates $A = (0,0)$ and $B = (c(x), 0)$. We can adjust the other points to these two. Let a third point be C, with distances $dist(A,C) = b(x)$ and $dist(B,C) = a(x)$. Let $C = (n(x), m(x))$. Now $A_{ABC}(x) = \frac{c(x)m(x)}{2}$, where $A_{ABC}(x)$ is the area of the ABC triangle. By that: $m(x) = \frac{2A_{ABC}(x)}{c(x)}$. Since we can compute the area using Heron's formula, the area can be written in the usual $l(x)\sqrt{k(x)}$ form, so:

$$m(x) = \frac{2l(x)\sqrt{k(x)}}{c(x)},$$

which is almost the desired form. Let $\angle BAC = \alpha(x)$. By the cosine rule: $\cos\alpha(x) = \frac{b^2(x)+c^2(x)-a^2(x)}{2b(x)c(x)}$, and since $\cos\alpha(x) = \frac{n(x)}{b(x)}$:

$$n(x) = \frac{b^2(x) + c^2(x) - a^2(x)}{2c(x)}$$

Now $C = (n(x), m(x)) = (\frac{b^2(x)+c^2(x)-a^2(x)}{2c(x)}, \frac{2l(x)\sqrt{k(x)}}{c(x)})$. Since $c(x)$ is the same for every point, we can multiply the distances by $c(x)$ and get the desired form. □

Remark. That means, the problem of finding integer point sets with characteristic 1 is the same as finding an integer point set where the points have integer coordinates, since the $k(x)$ used above is the characteristic.

4 Further research

In the next semesters my aim is to examine that if we fix a characteristic, then how can we generate parametric integer point sets, and hopefully I find new parametric constructions that may generate new integer point sets.

Also I plan to creating new parametrized sets using the method of uniting smaller ones.

References

[1] Anning, N. H., and Erdős, P. Integral distances. *Bull. Am. Math. Soc. 51, 598–600* (1945).

[2] Kreisel, T., and Kurz, S. There are integral heptagons, no three points on a line, no four on a circle. *Discrete & Computational Geometry 39* (05 2008).

[3] Kurz, S., Noll, L. C., Rathbun, R., and Simmons, C. Constructing 7-clusters. *Serdica Journal of Computing 8*, 1 (2014), 47–70.

[4] Minh, T. X. Rational heptagon. *Unpublished result*.

[5] Ulam, S. M. A collection of mathematical problems. *Interscience Publishers, New York and London* (1960).