Distribution-free Prediction and Confidence Regions for Open- and Closed-loop Stochastic Systems

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2025/01/09

Original Problem Setting

- Consider a linear regression system:

$$\left.\begin{aligned} \phi_1^T \theta^* + N_1 &= Y_1 \\ \phi_2^T \theta^* + N_2 &= Y_2 \\ &\dots \\ \phi_n^T \theta^* + N_n &= Y_n \end{aligned}\right\} \Phi^T \theta^* + N = Y$$

- The Least-Squares Estimate (LSE) of θ^*:

$$\hat{\theta} = (\Phi^T \Phi)^{-1} \Phi^T Y$$

- We aim to build non-asymptotic distribution-free confidence regions around the point estimate

Main Idea of SPS [1]

- Find $\hat{\theta}$, the root of the normal equation:

$$0 = \sum_{t=1}^{n} \phi_t (Y_t - \phi_t^T \theta) = \sum_{t=1}^{n} \phi_t \phi_t^T (\theta^* - \theta) + \sum_{t=1}^{n} \phi_t N_t = H_0(\theta)$$

- Perturb the signs of the prediction errors:

$$H_i(\theta) = \sum_{t=1}^{n} \alpha_{i,t} \phi_t \phi_t^T (\theta^* - \theta) + \sum_{t=1}^{n} \alpha_{i,t} \phi_t N_t$$

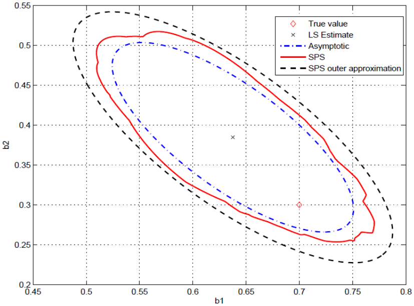
- Order them in some measure and define a subset over which θ is accepted

Remark

If θ is "close" to θ^*, the prediction error is "close" to the noise term so with symmetrically distributed noise the perturbed ones' distribution should remain the same.

Outer Approximation

- To get regions that are easier to calculate: ellipsoidal outer approximation
- It leads to $m-1$ convex minimization problem and the qth largest optimum gives the proper radius of the ellipsoid in the parameter space ($p = 1 - \frac{m}{q}$)
- To the function space: with ellipsoid transfromation



[1]

Generalization of SPS for Stochastic Systems

Assumptions we would like to relax:

exogenous regressors (independent from the noise terms)	$\Longleftrightarrow$	endogenous regressors
scalar valued data	$\Longleftrightarrow$	vector valued data

Scalar ARX Systems and Instrumental Variables

- Scalar ARX problem:

$$Y_t = \sum_{i=1}^{d_1} a_i^* Y_{t-i} + \sum_{i=1}^{d_2} b_i^* U_{t-i} + N_t \iff Y_t = \Phi_t^T \theta^* + N_t$$

- To handle the endogenous regressors - instrumental variables (ψ_t):
  1. must be correlated with the regressors
  2. cannot be correlated with the noise terms
- How to generate IVs:
  1. use only previous inputs ($\psi_t = [U_{t-1}, U_t]$)
  2. with least-squares estimation ($\psi_t = [\hat{Y}_t, U_t]$)
- The IV-estimate:

$$\hat{\theta}_{IV} = (\Psi^T \Phi)^{-1} \Psi^T Y$$

IV-SPS for Open-loop ARX Systems [5]

endogenous regressors, symmetrically distributed noise, scalar valued data

- Find $\hat{\theta}_{IV}$, the root of the normal equation:

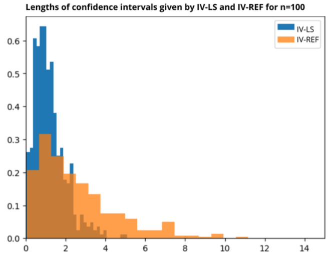
$$0 = \sum_{t=1}^{n} \psi_t (Y_t - \phi_t^T \theta) = \sum_{t=1}^{n} \psi_t \phi_t^T (\theta^* - \theta) + \sum_{t=1}^{n} \psi_t N_t = H_0(\theta)$$

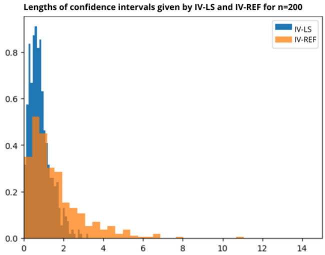
- Perturb the signs of the prediction errors:

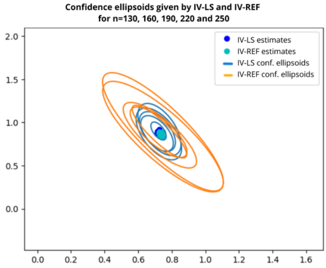
$$H_i(\theta) = \sum_{t=1}^{n} \alpha_{i,t} \psi_t \phi_t^T (\theta^* - \theta) + \sum_{t=1}^{n} \alpha_{i,t} \psi_t N_t$$

- Order them in some measure and define a subset over which θ is accepted
- Outer approximating ellipsoid in the parameter space → confidence interval in the state space

Performance of IV-Generating Methods







$$Y_t = a^* Y_{t-1} + b^* U_{t-1} + N_t$$
$$U_t = c^* U_{t-1} + V_t,$$

$$a^* = 0.75$$
$$b^* = 1$$
$$c^* = 0.75$$
$$p = 0.9$$

Vector Valued ARX Systems and Instrumental Variables - Open-loop Case

- Vector valued ARX problem
  (Y_t, U_t, N_t are vectors, A^*, B^* are matrices):

$$Y_t = \sum_{i=1}^{d_1} A_i^* Y_{t-i} + \sum_{i=1}^{d_2} B_i^* U_{t-i} + N_t \Longleftrightarrow Y = \Phi\Theta^* + N$$

- The IV-estimate:

$$\hat{\Theta}_{IV} = (\Psi^T \Phi)^{-1} \Psi^T Y,$$

  matrix valued!

MIV-SPS for Vector Valued ARX Systems [4]

endogenous regressors, symmetrically distributed noise, vector valued data

- Find $\hat{\Theta}_{IV}$, the root of the normal equation:

$$0 = \Psi^T(Y - \Phi\Theta) = \Psi^T\Phi(\Theta^* - \Theta) + \Psi^T N = H_0(\Theta)$$

- Perturb the signs of the prediction errors:

$$H_i(\Theta) = \Psi^T W_i \Phi(\Theta^* - \Theta) + \Psi^T W_i N,$$

  where $W_i = \text{diag}(\alpha_i)$

- Order them in some measure (Frobenius-norm) and define a subset over which Θ is accepted
- Outer approximating ellipsoid in the matrix-valued parameter space $\rightarrow$ confidence ellipsoid in the state space

Further Generalization of SPS

Assumptions we would like to relax:

open-loop system	$\Longleftrightarrow$	closed-loop system
ARX system	$\Longleftrightarrow$	ARMAX system

Vector Valued ARX Systems and Instrumental Variables - Closed-loop case [2]

- Vector valued ARX problem (direct approach) (Y_t, U_t, N_t, R_t are vectors, A^*, B^* are matrices):

$$Y_t = A_i^* Y_{t-1} + B_i^* U_{t-1} + N_t, \Longleftrightarrow Y = \Phi\Theta^* + N$$

$$U_t = FY_t + GR_t$$

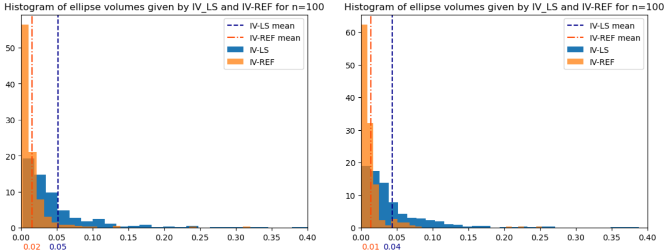
- Reformed as an open-loop system (indirect approach):

$$Y_t = (A^* + B^* F) Y_{t-1} + B^* G R_{t-1} + N_t \Longleftrightarrow Y = \Phi\Theta^* + N$$

- The IVs to use: $\psi_t = [R_{t-1}, R_t], \psi_t = [\hat{Y}_t, R_t]$
- The IV-estimate:

$$\hat{\Theta}_{IV} = (\Psi^T \Phi)^{-1} \Psi^T Y$$

Performance of IV-Generating Methods with Direct and Indirect Approaches



$$Y_t = A^* Y_{t-1} + B^* U_{t-1} + N_t$$
$$U_t = \epsilon F Y_t + (1-\epsilon) R_t,$$
A is a stable matrix,
elements in B $\sim U([1, 10])$,
F is the optimal LQR controller with $\epsilon = 1$, $p = 0.9$

ARMAX Systems and Prediction Error Method - Limitations of IV-Method [3]

- In more general problems no analytical solution $\rightarrow$ IV-method is not working for system identification
- **PEM**:

$$\hat{\theta}_{\text{PEM}} = \operatorname{argmin} \sum_{t=1}^{n} \hat{N}_t^2(\theta),$$

  can be found iteratively using methods such as the Newton method.
- ARMAX system (G, H, F and L are transfer functions):

$$Y_t = G(z^{-1};\theta^*)U_t + H(z^{-1};\theta^*)N_t,$$
$$U_t = -F(z^{-1};\theta^*)Y_t + L(z^{-1};\theta^*)R_t.$$

- SPS regions are build around $\hat{\theta}_{\text{PEM}}$, with alternative trajectories with the perturbed error terms

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Thank You for Your Attention!