Optimal transport and the pancake cut

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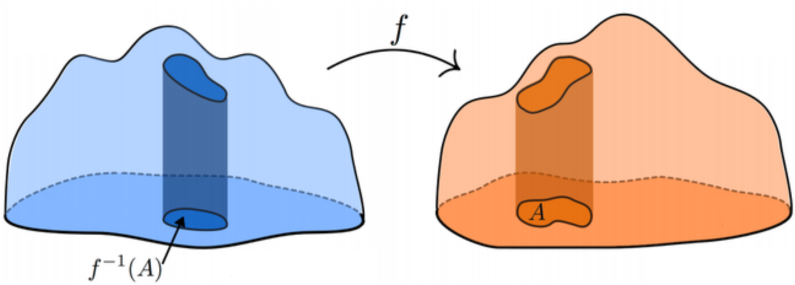
Optimal transport

Goal: Find a mass maintaining transport map from one measure μ to another ν, that minimizes a given cost function.



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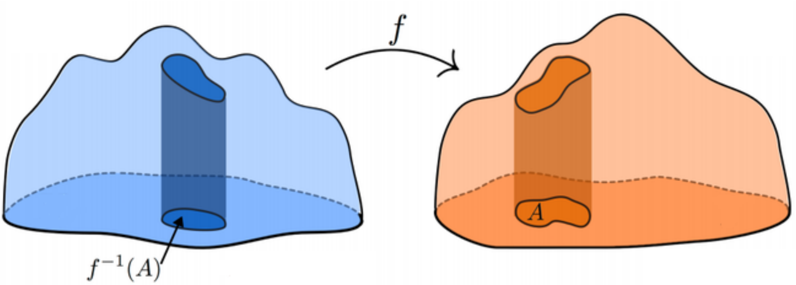

The discrete version of this is a maximum weight total matching
$\Rightarrow$ Hungarian method.

Discrete Kantorovich

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Mass preserving: In $\Pi(A, B)$ the marginals are correct

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Constraints:

$$\sum_{i=1}^{n} \Pi_{ij} = \mu_j, \; \sum_{j=1}^{m} \Pi_{ij} = \nu_i, \; \Pi_{ij} \geq 0$$

Objective function:

$$\min \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} \Pi_{ij}$$

Sinkhorn's algorithm

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New use [2]: Rescales a matrix step by step to make the marginals match.

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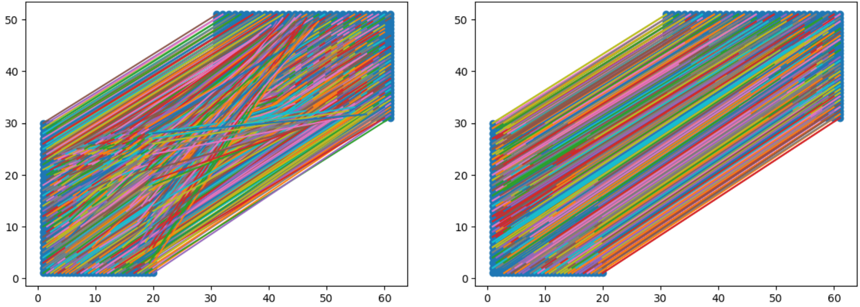
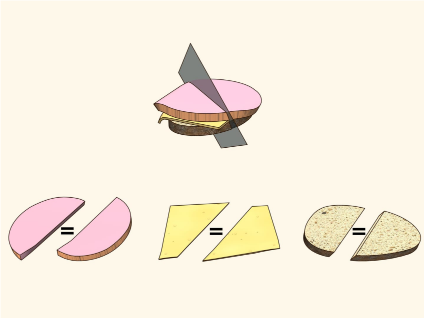
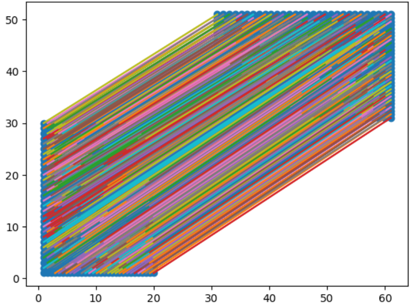
Unfortunately this solves a relaxation of the problem, the entropic regularization.



Pancake cut

The pancake problem is the two dimensional ham sandwich problem.

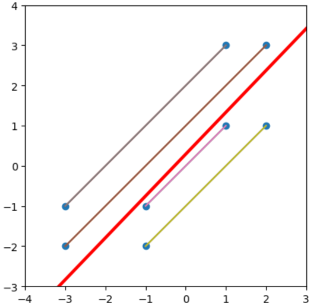


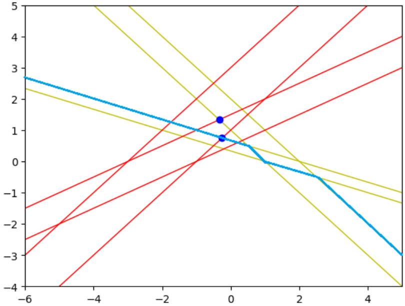


Megiddo's algorithm [3]

Goal: Bisect two sets of points in a way to halve both.

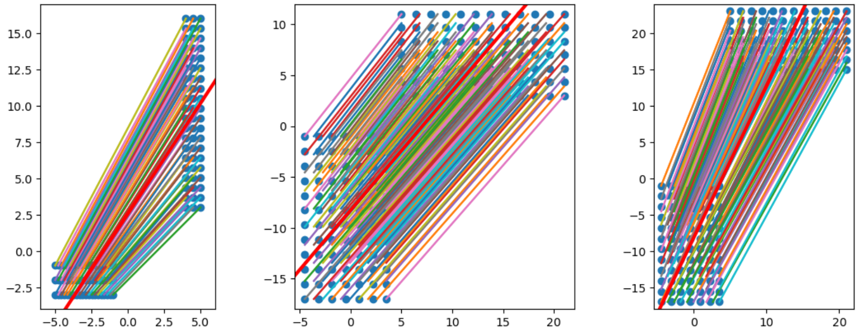
- **Trick:** conversion to lines $y = \frac{a_i}{bi} + \frac{-1}{b_i}x$
- **New goal:** The point where the two median functions intersect





Thank you for your attention!

It looks promising, next step is to evaluate these findings.



References

- Sinkhorn, Richard, & Knopp, Paul (1967): Concerning nonnegative matrices and doubly stochastic matrices. – Pacific J. Math. 21, 343–348.
- Cuturi, Marco (2013): Sinkhorn distances: Lightspeed computation of optimal transport. – Advances in neural information processing systems. 2292–2300.
- Megiddo, Nimrod (1985): Partitioning with two Lines in the Plane. – Journal of algorithms 6. 430-433
- Zhang, Hengning: "Megiddo's O(n) Linear Programming Visualization" https://hengningzhang.com/posts/megiddo/
- Transport figure from: https://freakonometrics.hypotheses.org/tag/cramer
- Ham sandwich figure from: https://en.etudes.ru/etudes/ham-sandwich-theorem/