# Comparison of iterative methods for discretized nonsymmetric elliptic problems

**Lados, Bálint István**

Supervisor: Karátson, János

Eötvös Loránd University
Department of Applied Analysis and Computational Mathematics

30 May, 2024

# Summary of my previous work

Let us consider the following nonsymmetric elliptic boundary value problem on $\Omega = [0,1]^2$:

$$\begin{cases} Lu := -\operatorname{div}(p\nabla u) + \mathbf{w}\cdot\nabla u = f, \\ u_{|\partial\Omega} = 0. \end{cases}$$

If the functions satisfy these conditions:

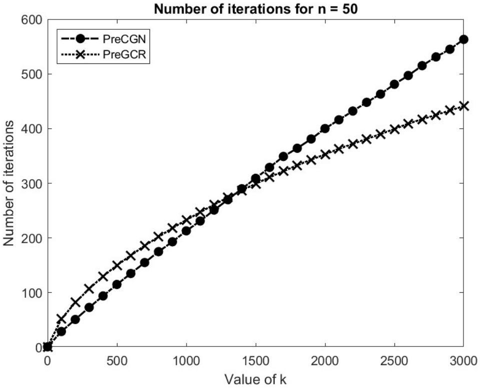
$$\begin{cases} p \in L^\infty(\Omega),\ p(x) \geq m > 0 \text{ (a.e. } x \in \Omega); \\ \mathbf{w} \in C^1(\overline{\Omega},\ \mathbb{R}^2),\ \operatorname{div}\mathbf{w} = 0, \end{cases}$$

then the PDE has a unique weak solution for any $f \in L^2(\Omega)$.

The exact solution of the PDE can be approximated with the finite difference method (**FDM**) or the finite element method (**FEM**).

# Previous results for the FDM

1. Introduce a set of discrete grid points in $\Omega$.
2. Approximate the partial derivatives in the grid points with finite differences, obtain a system of linear equations $Ax = b$.
3. Solve it with an iterative method: preconditioned **CGN** or **GCR**.



# Convection-dominated elliptic problems

Let us consider the following nonsymmetric elliptic boundary value problem on $\Omega = [0,1]^2$:

$$\begin{cases} -\varepsilon \Delta u + \mathbf{w} \cdot \nabla u = f \\ u|_{\partial \Omega} = 0 \end{cases}$$

where $\varepsilon > 0$ is constant and $\mathbf{w}$ satisfies the previous conditions.

- $-\varepsilon \Delta u$ models diffusion
- $\mathbf{w} \cdot \nabla u$ models convection
- $\varepsilon \approx 0$, which makes the convection term more dominant.

The solution is usually irregular (e.g. has large jumps), so the traditional numerical methods are not applicable here.

# The SDFEM / I.

We will use the *streamline diffusion finite element method* (**SDFEM**) to solve this special boundary value problem.

The weak solution of the problem is $u \in H_0^1(\Omega)$ for which

$$a(u,v) := \int_\Omega (\varepsilon \nabla u \cdot \nabla v + (\mathbf{w} \cdot \nabla u) v) = \int_\Omega fv, \quad \forall v \in H_0^1(\Omega).$$

**Problem:** The lower bound of $a(u,v)$ is $\varepsilon \approx 0$, which makes the constant $\frac{M}{\varepsilon}$ large in Céa's lemma slowing down the convergence.

**Solution:** Stabilise the coercivity bound by modifying a$(u,v)$:

$$a_{SD}(u,v) := \int_\Omega (\varepsilon \nabla u \cdot \nabla v + (\mathbf{w} \cdot \nabla u) v + \delta (\mathbf{w} \cdot \nabla u)(\mathbf{w} \cdot \nabla v)),$$

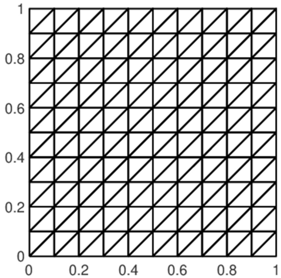
where $\delta > 0$ is a given parameter.

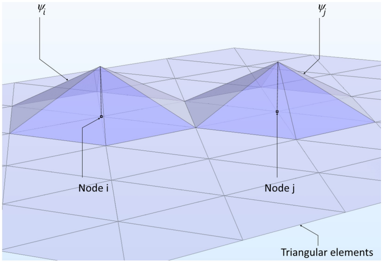
# The SDFEM / II.

We create a uniform triangular mesh in $\Omega$ with distance $h$ and construct the finite dimensional subspaces $V_h \subset H_0^1(\Omega)$ with Courant elements (piecewise linear functions on the triangles).

We are looking for the uniquely existing $u_h \in V_h$ for which

$$a_{SD}(u_h, v_h) = \int_\Omega f(v_h + \delta \mathbf{w} \cdot \nabla v_h) =: l(v_h), \quad \forall v_h \in V_h.$$



# Implementation of the SDFEM / I.

Let us define the basis functions $\{\phi_j\}_{j=1}^{N}$ in $V_h$ with the tent functions as shown above. We can express $u_h$ with them:

$$u_h = \sum_{j=1}^{N} c_j \phi_j$$

If we choose the test function $v_h := \phi_i$, then we obtain the equation $A\mathbf{c} = \mathbf{f}$, where $[A]_{i,j} = a_{SD}(\phi_j, \phi_i)$ and $\mathbf{f}_i = l(\phi_i)$.

The components of matrix $A$ and vector $\mathbf{f}$ can be approximated with the one-point quadrature:

$$\mathbf{f}_i = \int_\Omega f(\phi_i + \delta \mathbf{w} \cdot \nabla \phi_i) = \int_\Omega f\phi_i + \delta \int_\Omega f\mathbf{w} \cdot \nabla \phi_i \approx f(x_i, y_i) \int_\Omega \phi_i = f(x_i, y_i) \cdot h^2,$$

where the second term vanishes as $\int_\Omega \partial_x \phi_i = \int_\Omega \partial_y \phi_i = 0$.

# Implementation of the SDFEM / II.

The terms of $a_{SD}(\phi_j, \phi_i)$ can be approximated similarly, for example:

$$\int_\Omega (\mathbf{w} \cdot \nabla \phi_j)\phi_i = \int_\Omega (w_x \partial_x \phi_j)\phi_i + \int_\Omega (w_y \partial_y \phi_j)\phi_i \approx$$
$$w_x(x_i, y_j) \int_\Omega ((1,0) \cdot \nabla \phi_j)\phi_i + w_y(x_i, y_j) \int_\Omega ((0,1) \cdot \nabla \phi_j)\phi_i,$$

where the case of constant $\mathbf{w}$ can be calculated as in [3].

**Implementation:** Matlab.

# Convergence of the SDFEM

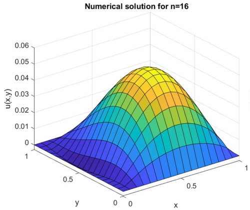
The numerical solution is expected to converge to the exact solution with an error of $O(h)$. Let us test the convergence:

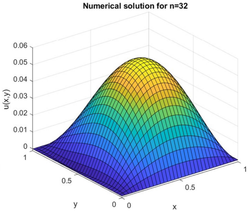
$$\varepsilon := 10^{-2}; \qquad p(x,y) := 1 + \frac{x^2+y^2}{2}; \qquad w(x,y) := (-y - \frac{1}{2}, x - \frac{1}{2})$$

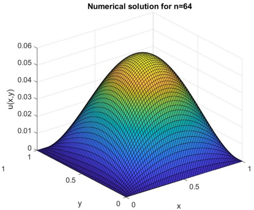
We choose $f$ such that the solution is $u(x,y) = x(1-x)y(1-y)$.

Doubling the grid density roughly halves the error:

| **n** | 8 | 16 | 32 | 64 | 128 | 256 |
|---|---|---|---|---|---|---|
| **e** | 0.0371 | 0.0240 | 0.0136 | 0.0071 | 0.0034 | 0.0015 |



# Comparison of the iterative methods

Let us consider the following two sets of convection-dominated elliptic problems depending on $k$ and $m$:

$$\begin{cases} -10^{-m}\Delta u + k(1,\,0)\cdot\nabla u = 1 \\ u|_{\partial\Omega} = 0 \end{cases} \qquad \begin{cases} -10^{-m}\Delta u + k(-y-\frac{1}{2},\, x-\frac{1}{2})\cdot\nabla u = 1 \\ u|_{\partial\Omega} = 0 \end{cases}$$

We can perform SDFEM discretisation on these problems.

**Problem:** We need to solve the system of equations $A\mathbf{c} = \mathbf{f}$.

**Question:** How does the number of iterative steps change for the preconditioned CGN and GCR algorithms as we decrease $\varepsilon$ and increase the norm of the vector field $\mathbf{w}$?

- The SDFEM parameter $\delta := h$;
- The preconditioner matrix $S := \frac{A+A^T}{2}$;
- The iterative methods run until the error $\|r_n\|_S < 10^{-5}$.

# Increasing the norm: *m* is fixed, *k* is varied

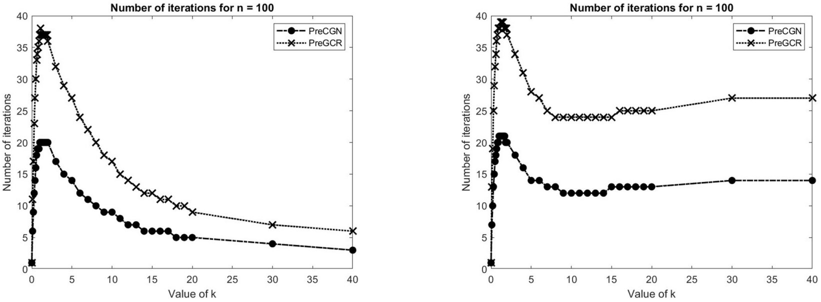



Figure: Number of iterative steps taken by the preconditioned CGN and GCR algorithms to solve $A\mathbf{c} = \mathbf{f}$ when $m = 2$ is fixed and $k$ is varied.

# Changing $\delta$, while $m$ is fixed, $k$ is varied

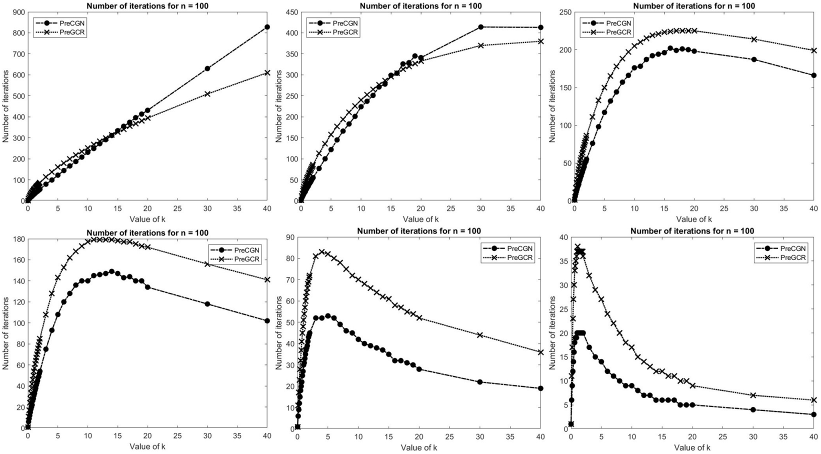



Figure: Number of iterative steps taken by the preconditioned CGN and GCR algorithms for $\delta = 0,\ 10^{-5},\ 5\cdot 10^{-5},\ 10^{-4},\ 10^{-3},\ 10^{-2}$ when $m = 2$ is fixed and $k$ is varied.

# Decreasing $\varepsilon$: $k$ is fixed, $m$ is varied

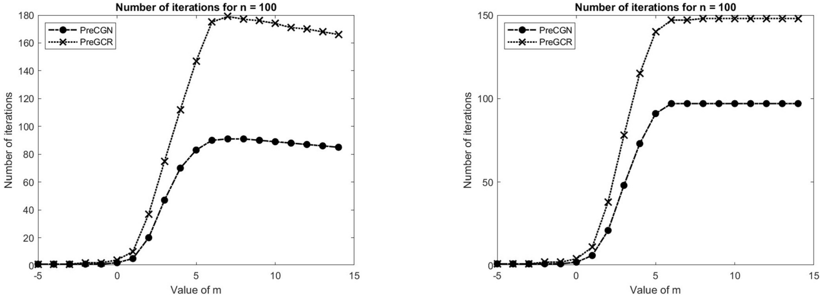



Figure: Number of iterative steps taken by the preconditioned CGN and GCR algorithms to solve $A\mathbf{c} = \mathbf{f}$ when $k = 1$ is fixed and $m$ is varied.

# Note on the shape of the previous graphs

The *streamline Poincaré–Friedrichs inequality* described in [1] gives an upper bound for the number of iterations independent of $\varepsilon$.

The characteristic curves of the vector field $\mathbf{w} = (1,0)$ can be parameterised with $\gamma_s(t) := (t,s)$, where $(s,t) \in \Omega = [0,1]^2$.

$$J_{\mathbf{w}}(s,t) = \left|\det \begin{pmatrix} \partial_s(t) & \partial_t(t) \\ \partial_s(s) & \partial_t(s) \end{pmatrix}\right| = \left|\det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right| = |-1| = 1$$

This is bounded from below and above by $\mu = \tilde{\mu} := 1$.

$$C_{\mathbf{w}} = \mathrm{diam}(\Omega) \cdot \sqrt{\tilde{\mu}/\mu} = \mathrm{diam}(\Omega) = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

In case of the CGN method, the bound is the following:

$$\left(\frac{\|r_n\|_S}{\|r_0\|_S}\right)^{\frac{1}{n}} \leq 2^{\frac{1}{n}} \cdot \frac{C_{\mathbf{w}}}{C_{\mathbf{w}} + 2\delta}$$

# References

## Thank You for Your Attention!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