Random matrices, perturbations and their applications in statistics

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Introduction, motivations

- Importance of examining data and covariance.
- How can we describe matrices of data effectively?
- Perturbation problem: how can we model noisy observation.
- Application of perturbation problem with examples.

- First singular value and vector of A.
- $\sigma_1 := \max_{|v|=1} |Av| \qquad \text{and} \qquad v_1 := \operatorname{argmax}_{|v|=1} |Av|.$
- By induction, let σ_i be the i-th singular value of matrix A (for $i = 2 \dots r$) and let denote the i-th singular vector of matrix A by v_i, if

$$\sigma_i = \max_{v:|v|=1, v \perp v_1, \dots v_{i-1}} |Av| \quad \text{and} \quad v_i = \operatorname{argmax}_{|v|=1, v \perp v_1, v_2 \dots v_{i-1}} |Av|.$$

Equivalent definition for singular vectors and values

- We assume that $A \in \mathbb{R}^{d_1 \times d_2}$ with rank n. There exists only one $U \in \mathbb{R}^{d_1 \times n}$ and $V \in \mathbb{R}^{d_2 \times n}$ orthogonal and $\Sigma \in \mathbb{R}^{n \times n}$ positive definit diagonal matrix with, that

$$A = U\Sigma V^T = \sum_{i=1}^{n} \sigma_i u_i v_i^T,$$

.

- Here $\Sigma_{11} = \sigma_1 > \Sigma_{22} = \sigma_2 > \ldots \Sigma_{nn} = \sigma_n > 0$,
- u_i and v_i are the i-th columns of U and V,
- and σ_i and v_i are the i-th singular values and vector of matrix A.

First simulated theorem-O'Rourke, S.; Vu, V.; Wang, K.:

I simulated the theorem of this slide with two diagrams.

- The matrix E is called Bernoulli matrix if

$$E = [E]_{i,j}, \qquad P(E_{i,j} = 1) := P(E_{i,j} = -1) := 0.5$$

  with independent coordinates.
- The theorem of O'Rourke, Vu, Wan with Bernoulli matrix: if A is data matrix with (low) rank r and E is a random Bernoulli matrix, then for every $\varepsilon > 0$ there exist constants $C, \delta_0 > 0$ such that if

$$\sigma_1 - \sigma_2 := \delta \geq \delta_0 \quad \text{and} \quad \sigma_1 \geq \max\{n, \sqrt{n} \cdot \delta\}.$$

  then with a probability at least $1 - \varepsilon$ the inequality

$$\boxed{\sin\left(< (v_1, v_1')\right) \leq C \cdot \frac{\sqrt{r}}{\delta}}$$

  fulfils (where $\sin\left(< (v_1, v_1')\right)$ is the sinus of the closed angle of the first singular vector of A and $A + E$).

First simulation

- A: I chose out three deterministic matrices with only four non zero elements, with corresponding δ and rank 2.

- E: I added 400 independent Bernoulli matrices E to them, so I got $400 \cdot 3$ new matrices $(A + E)$

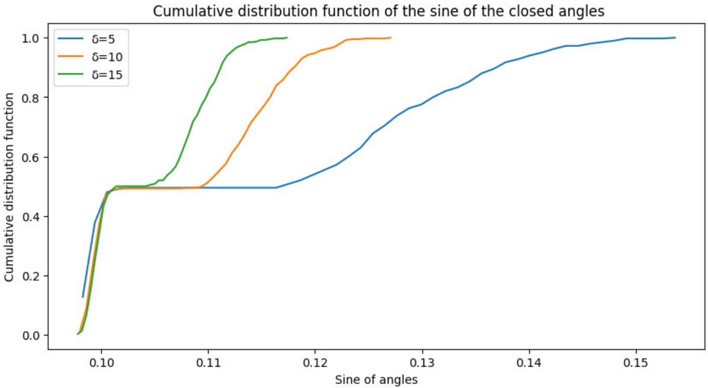

Figure: Sine of closed angles of the first singular vector of A and $A+E$ with a simple two-rank matrix A and Bernoulli matrix E.

First simulation-second diagram

- A: I chose one random matrix from Wishart distribution with rank 2.
- E: I added 100 independent Bernoulli matrices E to it, so I got 100 new matrices $(A + E)$

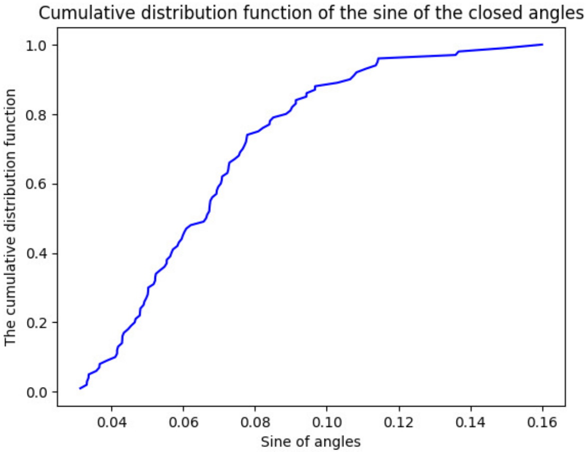

Figure: Sine of closed angles with Wishart matrix A and Bernoulli matrix E.

Second simulated theorem

- I simulated another theorem with two diagrams, too. Now I will define some notions:
- If $M = [M_{ij}] \in \mathbb{R}^{d_1 \times d_2}$ as follows:

$$\|M\|_{\max} := \max_{ij} |M_{ij}|, \qquad \|M\|_{\infty} := \max_{i} \sum_{j=1}^{d_2} |M_{ij}|.$$

- $\mu := \mu(V) := \frac{d_2}{n} \cdot \max_i \sum_{j=1}^{n} V_{ij}^2.$
- Let A_r be the the best rank-r approximation of A under the Forbenius norm:

$$A_r := \sum_{i=1}^{r} \sigma_i u_i v_i^T.$$

- $A' := A + E := U' \Sigma' V'^T.$

Second simulated theorem-Fan, Jianqing; Wang, Weichen; Zhong, Yiqiao:

- We suppose that $\delta > ||E||_2$ and $\sigma_r - \varepsilon = \Omega(r^3\mu^2||E||_\infty)$,
- where $\varepsilon = ||A - A_r||_\infty$.
- If A is symmetric and for any $i = 1, \dots r$ the interval $[\sigma_i - \delta, \sigma_i + \delta]$ does not contain any singular values of A other than σ_i, then

$$||V' - V||_{\max} = \mathcal{O}\left(\frac{r^4\mu^2||E||_\infty}{(\sigma_r - \varepsilon)\sqrt{n}} + \frac{r^{\frac{3}{2}}\mu^{\frac{1}{2}}||E||_\infty}{\delta\sqrt{n}}\right).$$

- I determined the singular value decomposition of $(A' = A + c \cdot E, A)$ for every $c \in [0, 100]$, and got (V', V) matrices and plotted the mean of the cases of $||V' - V||_{\max}$ depending on the function of c.

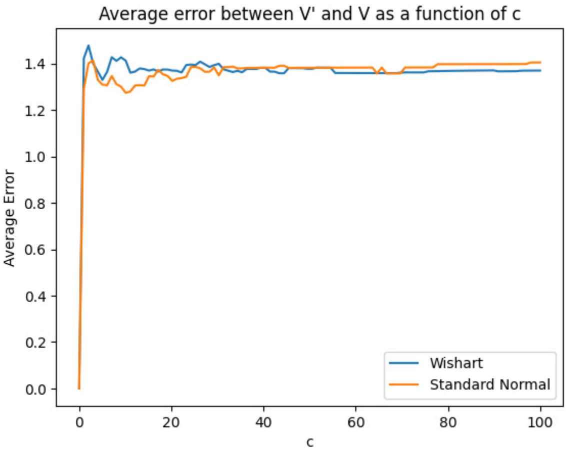

Figure: Simulation results for Theorem 1 with $0 \leq c \leq 100$.

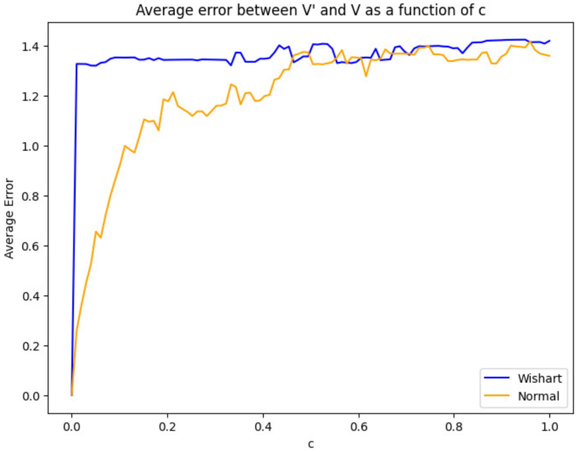

Figure: Simulation results for Theorem 1 with $0 \leq c \leq 1$.

Other read articles-curiosities

- I read two another articles this semester, I would like to summarize them shortly.
- In Chen, Y.; Cheng, Ch..; Fan, Y.: Asymmetry helps-strong estimation about the norm of the noise matrix. They could bound from below the norm of the noise matrix with the distance of the right and estimated leading eigenvalue.
- Benaych-Georges, F.; Nadakuditi, R. R.: the authors examined the convergence of the corresponding singular values of the low rank matrices and could give the limit in some special cases.

Future plans

- Deeper understanding of the applications of perturbed random matrices in statistics.
- To understand how the structure (distribution) of the matrix can affect the behaviour of its singular vectors.
- To survey the newest articles of the literature of random perturbed matrices.
- To apply the above to real data.

- Benaych-Georges, F.; Nadakuditi, R. R.: *The singular values and vectors of low rank perturbations of large rectangular random matrices*, J. Multivariate Anal. **111** (2012), 120–135.
- Blum, A.; Hopcroft, J.; Kannan, R.: *Foundations of Data Science*, Cambridge University Press, 2020.
- Chen, Y.; Cheng, Ch..; Fan, Y.: *Asymmetry helps: Eigenvalue and eigenvector analyses of asymmetrically perturbed low-rank matrices*, Annals of statistics, 2021, 49.1: 435.
- Fan, Jianqing; Wang, Weichen; Zhong, Yiqiao: *An l^∞ eigenvector perturbation bound and its application to robust covariance*, J. Mach. Learn. Res. 18 (2017), Paper No. 207, 42 pp.
- O'Rourke, S.; Vu, V.; Wang, K.: *Random perturbation of low rank matrices: Improving classical bounds*, Linear Algebra and its Applications **540** (2016), 26–59.

Thank you for your attention!