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Higher Connectivities of Matroids

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1 Introduction

Tutte's goal was to define higher connectivities on matroids in such a way that the connectivity number is equal for the matroid and its dual, or equal for a graph and its graphic matroid.

A matroid is *2-connected* if and only if for every pair of elements in the ground set, there exists a circuit containing both elements. Using the submodular property of the rank function, it can be observed that $r(X) + r(E - X) \geq r(E)$ where E is the edge set of the graph and the ground set of the matroid. Furthermore, X is a separator exactly when $r(X) + r(E - X) - r(E) = 0$. A matroid is $2-$*connected* if and only if it has no separator, which means $r(X) + r(E - X) - r(E) \geq 1$ for all $X \subset E$.

Generalizing this idea, Tutte defined the *connectivity-function* for all $X \subset E$ as:

$$\lambda_{\mathcal{M}}(X) := r(X) + r(E - X) - r(E).$$

Because of symmetry, $\lambda(X) = \lambda(E - X)$, and by knowing that $r^*(X) = |X| + r(E - X) - r(E)$, the following can be seen as well: $\lambda_{\mathcal{M}}(X) = r(X) + r^*(X) - |X|$, consequently, $\lambda_{\mathcal{M}}(X) = \lambda_{\mathcal{M}^*}(X)$.

If $\lambda(X) < k$, then X and $(X, E - X)$ are called *k-separating*. If X is k-separating and $min\{|X|, |E - X|\} \geq k$ then it is a *k-separation* with sides X and $E - X$. A matroid $\mathcal{M}$ is *n-connected* when it has no k-separation, for all $k < n$.

With these definitions, follows instantly that $\mathcal{M}$ is n-connected if and only if $\mathcal{M}^*$ is n-connected.

Furthermore, Tutte proved the following:

Theorem 1.1. *[1] Let G be a graph having no isolated vertices.*

- *If $|V(G)| \geq 3$, then $\mathcal{M}(G)$ is 2-connected $\Leftrightarrow$ G is 2-connected and loopless.*
- *If $|E(G)| \geq 4$, then $\mathcal{M}(G)$ is 3-connected $\Leftrightarrow$ G is 3-connected and simple.*

In other words for 2-connectedness, 1-circles are banned; for 3-connectedness, 2-circles are banned as well. For 4-connectedness 3-circles or triangles would also have to be banned but that would lead to a significant violation of generality. Instead, Tutte introduced a new concept.

2 Vertical connectivity

Let k be a positive integer. A partition (X, Y) of the ground set is a *vertical k-separation* of $\mathcal{M}$ if $\lambda(X) < k$ and $min\{r(X), r(Y)\} \geq k$.

Remark 2.1. *[1] If $\mathcal{M}$ has a vertical k-separation (X, Y), then $k+k-r(E) \leq r(X)+r(Y)-r(E) \leq k-1$, hence $k \leq r(E)-1$ and $\max\{r(X), r(Y)\} \leq r(E)-1$.*

Therefore, if (X, Y) is a vertical k-separation, then X and Y contain cocircuits of $\mathcal{M}$ because $r^*(X) = |X| + r(E-X) - r(E) = |X| + r(Y) - r(E) \leq |X| + (r(E)-1) - r(E) = |X| - 1$, which leads to X being not independent in $\mathcal{M}^*$, meaning X contains a circuit of $\mathcal{M}^*$.

Conversely, when $\mathcal{M}$ has two disjoint cocircuits, then it is straightforward to check whether $\mathcal{M}$ has a vertical k-separation for some $k \leq r(E)-1$.

If $\mathcal{M}$ has two disjoint cocircuits then its *vertical connectivity number* $\kappa(\mathcal{M}) := min(j)$, where $\mathcal{M}$ has no vertical k-separation for all $k < j$, otherwise $\kappa(\mathcal{M}) := r(\mathcal{M})$. A matroid $\mathcal{M}$ is *vertically n-connected* if $2 \leq \kappa(\mathcal{M}) \leq n$. Now the main statement can be formulated:

Theorem 2.2. *[1] G is a connected graph. Then $\kappa(\mathcal{M}(G)) = \kappa(G)$.*

3 Outlook

Building on these fundamentals, I started to check the vertical connectivity number of other matroids.

3.1 Uniform matroids

Claim 3.1. *[1] For the uniform matroid $U_{n,r}$, the vertical connectivity number is:*

$$\kappa(\mathcal{M}) = \begin{cases} n-r+1, & \text{if } n \leq 2r-2 \\ r, & \text{otherwise.} \end{cases}$$

3.2 Transversal matroids

I found two inequalities regarding the vertical connectivity number of transversal matroids. Let $G = (S, T; E)$ be a bipartite graph and $\mathcal{T}(G) = (S, I)$ the transversal matroid where the independent sets are those subsets of S which can be covered by matchings in G.

Let's observe that if G has a matching covering S, then the rank of the matroid is $|S|$ and all subsets are independent. For any $X \subset S : \lambda(X) = r(X)+r(S-X)-r(S) = |X|+|S-X|-|S| = 0$, so $\kappa(\mathcal{T}(G)) = 1$. Moreover, for any M matching and any X stands: $r(X) \leq |X|, r(S-X) \leq |S-X|, r(S) \geq |M|$. Hence, $\lambda(X) \leq |S| - |M|$. Therefore, the following claim holds:

Claim 3.2. $\kappa(\mathcal{T}(G)) \leq |S| - \nu(G) + 1$, *where $\nu(G)$ is the size of the maximum matching.*

If G is connected, then another bound is the following:

Claim 3.3. $\kappa(\mathcal{T}(G)) \leq \kappa(\mathcal{M}(G)) + 1 = \kappa(G) + 1$.

Proof. Let $\kappa(G) = n$. Then there exists a minimal vertex-cut V' with size n. V' can be positioned in three ways: contained in T, contained in S or intersecting both, as shown in Figure 1. In all cases, let G_1 be one component of $G - V'$, and let X be the vertices in $G_1 \cap S$.

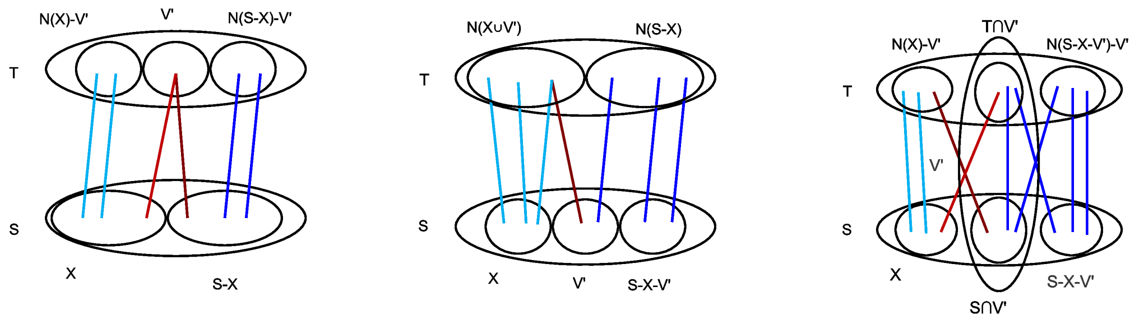


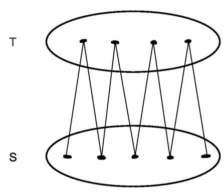
Figure 1: The cases of positioning V'

	V' ⊆ T case	V' ⊆ S case	V' intersects both
light blue	X, N(X)-V'	X, N(X∪V')	X, N(X)-V'
light red	X, V'	-	X, T∩V'
dark blue	S-X, N(S-X)-V'	S-X, N(S-X)	S-X, T-(N(X)-V')
dark red	S-X, V'	V', N(X∪V')	S∩V', N(X)-V'

The positioning of the different color classes depending on the cases of V'

The light blue and red edges represent a maximal matching on the subset X, while the dark blue and red edges represent a maximal matching on the complement set $S - X$. The light blue and dark blue edges are a matching on S, but the red edges can be incident on the same vertices with some already used vertices. In every case at least one end-vertex of a red edge is in V'. If a vertex in V' is covered by two red edges, one of them can be used in the united matching. Thus, the number of extra edges can be bounded above by $|V'| = n$. Consequently, $\lambda(X) = r(X) + r(S - X) - r(S) \leq n < n + 1$. □

In the example, the graph is 1-connected and the maximal matching has a size of 4. For any $X \subset S : r(X) + r(S - X) = 5$, hence, $\lambda(X) = 5 - 4 = 1$. Therefore, $\kappa(\mathcal{T}(G)) = 2 = 1 + 1 = \kappa(G) + 1 = |S| - \nu(G) + 1$. Thus, this upper bound cannot be reduced in the two cases.



If a matching covering S exists, then $\kappa(\mathcal{T}(G)) = 0$, as discussed above. If G is n-connected for $n > 0$, then $\kappa(\mathcal{T}(G)) < \kappa(G) + 1$. Therefore, the statement $\kappa(\mathcal{T}(G)) = \kappa(G) + 1$ cannot generally hold true.

References

[1] James G. Oxley. “Matroid Theory”. In: *Oxford University Press* (1992).