Higher Connectivities of Matroids

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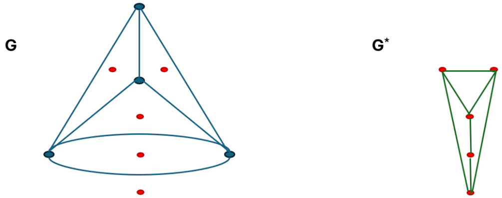
Motivation

Define higher connectivities on matroids such that

- the connectivity number is equal for the matroid and its dual or
- equal for a graph and its graphic matroid.



Connectivity function

Definition

The *connectivity-function* for all $X \subset E$:

$$\lambda_{\mathcal{M}}(X) := r(X) + r(E - X) - r(E).$$

Definition

Let k be a positive integer. A partition (X, Y) of the ground set is a *k-separation* of $\mathcal{M}$ if

- $\lambda(X) < k$ and
- $\min\{|X|, |Y|\} \geq k$.

Definition

A matroid $\mathcal{M}$ is *n-connected* when it has no k-separation, for all $k < n$.

Results with k-separation

Claim

$\mathcal{M}$ is *n*-connected if and only if $\mathcal{M}^*$ is *n*-connected.

Theorem (Tutte)

Let G be a graph with no isolated vertices.

- *If $|V(G)| \geq 3$, then $\mathcal{M}(G)$ is 2-connected if and only if G is 2-connected and loopless.*
- *If $|E(G)| \geq 4$, then $\mathcal{M}(G)$ is 3-connected if and only if G is 3-connected and simple.*

Vertical connectivity

Definition

Let k be a positive integer. A partition (X, Y) of the ground set is a *vertical k-separation* of $\mathcal{M}$ if

- $\lambda(X) < k$ and
- $\min\{\mathbf{r}(X), \mathbf{r}(Y)\} \geq k$.

Definition

Let the *vertical connectivity number* of $\mathcal{M}$ be

$$\kappa(\mathcal{M}) := \begin{cases} \min\{j : \mathcal{M} \text{ has no vertical } k\text{-separation for all } k < j\} \\ \qquad\qquad \text{if } \mathcal{M} \text{ has two disjoint cocircuits,} \\ r(\mathcal{M}), \ \text{otherwise.} \end{cases}$$

Results with vertical k-separation

Graphic matroids

Theorem

Let G be a connected graph. Then $\kappa(\mathcal{M}(G)) = \kappa(G)$.

Other matroids

Uniform matroids

Claim

$$\kappa(U_{n,r}) = \begin{cases} n-r+1, & \text{if } n \leq 2r-2 \\ r, & \text{otherwise.} \end{cases}$$

Transversal matroids

Claim

$\kappa(\mathcal{T}(G)) \leq |S| - \nu(G) + 1$, where $\nu(G)$ is the size of the maximum matching.

Claim

If G is connected, then $\kappa(\mathcal{T}(G)) \leq \kappa(\mathcal{M}(G)) + 1 = \kappa(G) + 1$.

Bibliograhy

1. James G. Oxley (1992): Matroid Theory, *Oxford University Press*

Thank you for your attention!