Matroid parameters for fixed-parameter tractability

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Fixed parameter tractability

Definition

A *parameterized problem* is a language $L \subset \Sigma^* \times \mathbb{N}$, where Σ is a fixed, finite alphabet and Σ^* denotes the set of all words gained from Σ. For an instance $(x, k) \in \Sigma^* \times \mathbb{N}$, k is called the *parameter*.

Definition

A parameterized problem $L \subset \Sigma^* \times \mathbb{N}$ is called *fixed-parameter tractable* (FPT) if there exists an algorithm $\mathscr{A}$ (called a fixed-parameter algorithm), a computable function $f : \mathbb{N} \to \mathbb{N}$, and a constant c such that, given $(x, k) \in \Sigma^* \times \mathbb{N}$, the algorithm $\mathscr{A}$ correctly decides whether $(x, k) \in L$ in time bounded by $f(k) \cdot |(x, k)|^c$. The complexity class containing all fixed-parameter tractable problems is called *FPT*.

Matroids and connectivity functions

Definition

A set-system $M = (S, \mathscr{F})$ is called a **matroid** if it satisfies the following properties, called **independence axioms**.

- $\emptyset \in \mathscr{F}$.
- If $X \subseteq Y \in \mathscr{F}$, then $X \in \mathscr{F}$.
- For every subset $X \subseteq S$, the maximal subsets of X which are in $\mathscr{F}$ have the same cardinality.

Matroids and connectivity functions

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$
$$\begin{matrix} a & b & c & d & e & f & g & h \end{matrix}$$

Figure 1: Linear matroid

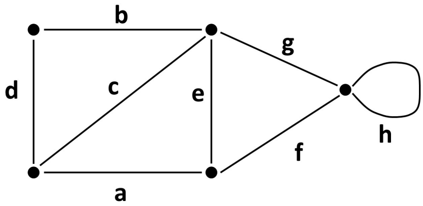


Figure 2: Graphic matroid

Figure 3: Figure 1 and 2 are two representations of the same matroid

Matroids and connectivity functions

Definition

$\lambda : 2^S \to \mathbb{Z}$ is a **connectivity function** if it satisfies the following three properties.

- $\lambda(\emptyset) = 0$,
- $\lambda(X) = \lambda(S \setminus X)\ \forall X \subseteq S$, symmetry,
- $\lambda(X) + \lambda(Y) \geq \lambda(X \cap Y) + \lambda(X \cup Y)\ \forall X, Y \subseteq S$, submodularity.

Graph $G = (V, E)$: for $X \subseteq E$, $\lambda_G(X)$ is the number of vertices, which have edges from both X and $E \setminus X$ sets.

Matroid $M = (S, \mathscr{F})$: $\lambda_M(X) = r_M(X) + r_M(S \setminus X) - r_M(S)$

Branch-width

Definition

A **branch-decomposition** of H (where H can be an edge set of a graph, hypergraph, the domain of a function or a matroid) is a (T, L) pair, where T is a sub-cubic tree (all nodes have at most 3 neighbours), and L is a bijection between the elements of H, and the leaves of T. (T, L) is a **partial branch-decomposition** if L is only surjective.

An edge $e \in T$, splits it into two connected components. This gives a partition (E_1, E_2) in H. Using this, we can specify the width of the decomposition, along with the branch-width.

Branch-width

Definition

For a connectivity function λ the **width of an edge** e is $\lambda(E_1) = \lambda(E_2)$, where (E_1, E_2) is the partition induced by e. For a graph $G = (V, E)$ it's $\lambda_G(E_1) = \lambda_G(E_2)$, where λ_G is the connectivity function of G. For a matroid M, it's $\omega_T(e) = \lambda_M(E_1) + 1 = \lambda_M(E_2) + 1$, where λ_M is the connectivity function of matroids.

Definition

The **width of T**, if T is a branch-decomposition, is the maximum edge-width for all $e \in T$. **Branch-width** is the minimum width over all branch-decompositions. Its notation for graphs, matroids and connectivity-functions is $\mathrm{bw}(G)$, $\mathrm{bw}(M)$ and $\mathrm{bw}(\lambda)$, respectively.

Branch-width

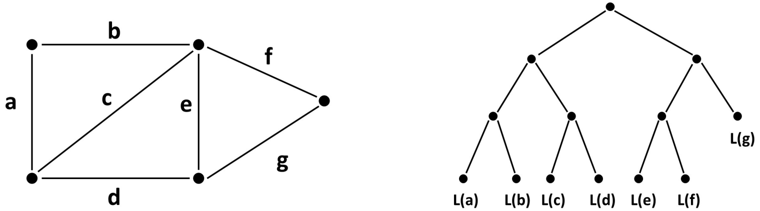


Figure 4: Example of a branch-decomposition. The graph on the left is viewed as a cycle matroid, the right picture shows an optimal branch-decomposition with $\mathrm{bw}(M) = 3$.

Branch-depth

Definition

Let S be a finite set of elements. A **depth-decomposition** of a connectivity function $\lambda : 2^S \to \mathbb{Z}$ is a (T, L) pair, where T is a tree with at least one internal node.

The **radius** of a (T, L) decomposition is the radius of the tree T. It is the smallest number r, so there exists a node with distance at most r from every node.

Branch-depth

Definition

Let (T, L) be a decomposition of a connectivity function λ. For an internal node $v \in V(T)$, the connected components of the graph $T \setminus \{v\}$ give a partition $\mathscr{P}_v$ on E by L. The **width of** v is defined to be $\lambda(\mathscr{P}_v)$, where $\lambda(\mathscr{P}_v) = \max_{\mathscr{P} \subseteq \mathscr{P}_v} \lambda_M \left(\bigcup_{X \in \mathscr{P}} X\right)$.
The **width of the decomposition** (T, L) is the maximum width of an internal node of T. We say that a decomposition (T, L) is a (k, r)-decomposition of λ if the width is at most k and the radius is at most r.
The **branch-depth** of λ, denoted by $\mathrm{bd}(\lambda)$, is the minimum k such that there exists a (k, k)-decomposition of λ.

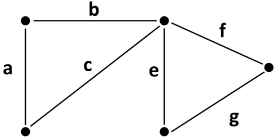
Depth-parameters

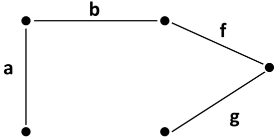
Deletion-depth, contraction-depth, contraction-deletion-depth.

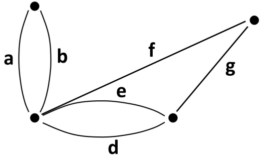
Definition

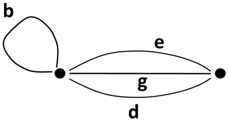
- If $E(M) = \emptyset$, then $\mathrm{dd}(M) = \mathrm{cd}(M) = \mathrm{cdd}(M) = 0$.
- If M is not connected, then $\mathrm{dd}(M)$, $\mathrm{cd}(M)$, $\mathrm{cdd}(M)$ is the maximum respective depth of the matroid's components.
- If M is connected, and $E(M) \neq \emptyset$, then:
  - $\mathrm{dd}(M) = 1 + \min_{e \in M}\{\mathrm{dd}(M \setminus e)\}$.
  - $\mathrm{cd}(M) = 1 + \min_{e \in M}\{\mathrm{cd}(M/e)\}$
  - $\mathrm{cdd}(M) = \min\{\min_{e \in M}\{\mathrm{dd}(M \setminus e)\}, \min_{e \in M}\{\mathrm{cd}(M/e)\}\}$

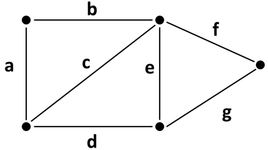
Depth-parameters











Depth-parameters

Definition

Deletion-decomposition tree:

- If M has a single element, the tree has a single vertex, labelled with the element.
- If M is disconnected the tree is obtained by merging the roots of the decomposition trees of the components of M.
- If M is connected, there exists an element $e \in M$ such that $\mathrm{dd}(M) = \mathrm{dd}(M \setminus e) + 1$. The tree is obtained by attaching the decomposition tree of $M \setminus e$ to a new vertex, label the edge by the deleted element e, and change the root to the newly added vertex.

  The **deletion-depth** is then the smallest height of the tree.

Depth-parameters

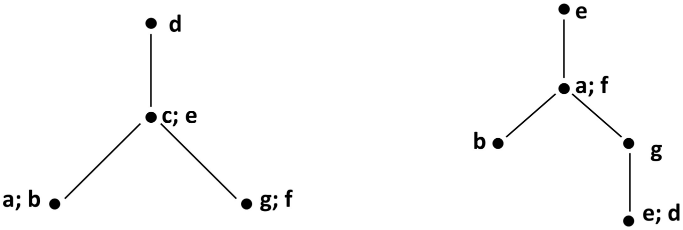


Figure 6: Example of deletion- and contraction-decompositions

Depth-parameters

Definition

Contraction*-depth decomposition is a pair (T, f), where T is a tree with $r(M)$ edges and f maps the elements to the leaves, so for every set of elements $X \subseteq S$, the number of edges in the rooted subtree induced by $f(X)$, denoted by $||T^*(X)||$, is at least $r(X)$.
Contraction*-depth is the minimum depth of a contraction*-depth-decomposition of M.

Depth-parameters

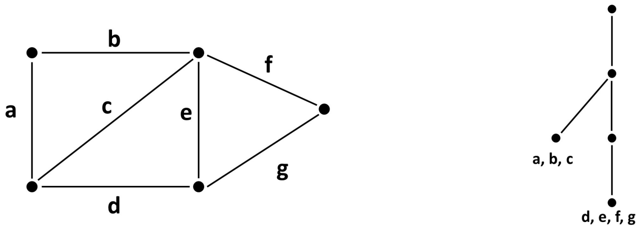


Figure 7: Example of a contraction*-decomposition. The consequence of this decomposition is that $\mathrm{c}^*\mathrm{d}(M) = 3$, since it is relatively easy to see, that no other decomposition with less depth would fulfill the rank requirements.

Depth-parameters

Definition

Contraction*-depth: For **representable** matroids.

- If M has a single element, then $\mathrm{c}^*\mathrm{d}(M) = r(M)$, which means it's either 0 or 1, depending on if the element is a loop or not.
- If M is not connected, then $\mathrm{c}^*\mathrm{d}(M)$ is the maximum contraction*-depth of a component of M.
- $\mathrm{c}^*\mathrm{d}(M) = 1 + min(M/K)$ factoring M over an arbitrary one-dimensional subspace.

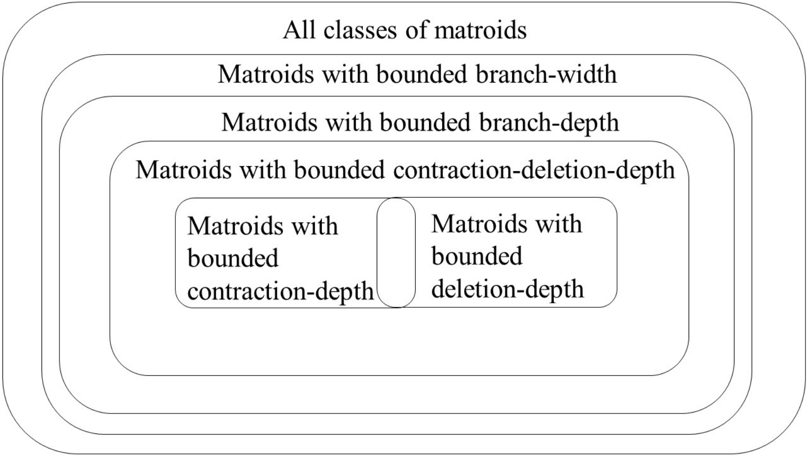
Depth-parameters

Contraction*-deletion-depth

Definition

- If $r(M) = 0$ then $\mathrm{c}^*\mathrm{dd}(M) = 0$.
- If $r(M) = 1$ then $\mathrm{c}^*\mathrm{dd}(M) = 1$.
- If M is disconnected, then $\mathrm{c}^*\mathrm{dd}(M)$ is the maximum contraction*-deletion-depth of components of M.
- If M is connected then $\mathrm{c}^*\mathrm{dd}(M)$ is the minimum contraction*-deletion-depth of $(M \setminus e)$ increased by 1.

Connections



Thank you for your attention!