Helly-type theorems and boxes

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Introduction

- If property A holds for any **subfamily** of a family of sets $\mathcal{F}$ that is of a **given finite size** h and property, then some property B holds for the whole **family** $\mathcal{F}$ of **arbitrary finite size** n
- Equivalent statement: If property B doesn't hold for $\mathcal{F}$, then A doesn't hold for some subfamily of size h.
- Helly number: h (minimal)

Helly-type theorems

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- **Helly's original statement**
  - convex sets
  - non-empty intersection
  - Helly number: $d + 1$

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  - convex sets
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- **Colorful Helly Theorem**
- **Quantitative Volume Theorem**
  - convex sets
  - lower bound on **volume** of intersection
  - Helly number: $2d$

Piercing Boxes

Definition: A set P **pierces** a family of sets $\mathcal{F}$ if for any set $S \in \mathcal{F}$ there is an element $p \in P$ such that $p \in S$. If $|P| = n$, then $\mathcal{F}$ is n-**pierceable**

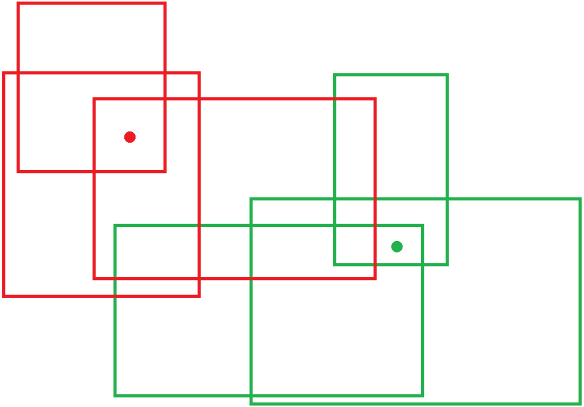

Figure: 2-piercing a family of 2-dimensional boxes

Piercing Boxes

- All Helly-type statements are proven!

Piercing Boxes

- All Helly-type statements are proven!
- **Theorem** (Danzer, Grünbaum). *If $h = h(d,n)$ is the smallest positive integer such that for any finite family $\mathcal{F}$ of axis-parallel boxes in $\mathbb{R}^d$ every h-tuple from $\mathcal{F}$ is n-pierceable implies that $\mathcal{F}$ is n-pierceable then following are the values of h:*

$$
\begin{aligned}
h(d,1) &= 2 \\
h(1,n) &= n+1 \\
h(d,2) &= \begin{cases} 3d & : \quad 2 \mid d \\ 3d-1 : & \quad 2 \nmid d \end{cases} \\
h(2,3) &= 16 \\
h(d,n) &= \aleph_0 \quad n \geq 3, (d,n) \neq (2,3)
\end{aligned}
$$

Punching holes into boxes

piercing and **volume** $\rightarrow$ punching holes

n-punching

Family of d-dimensional boxes $\mathcal{F}$ is n-punchable:

- $\exists$ A_1, A_2, ..., A_n boxes of volume 1 each
- Any box from $\mathcal{F}$ contains some A_i

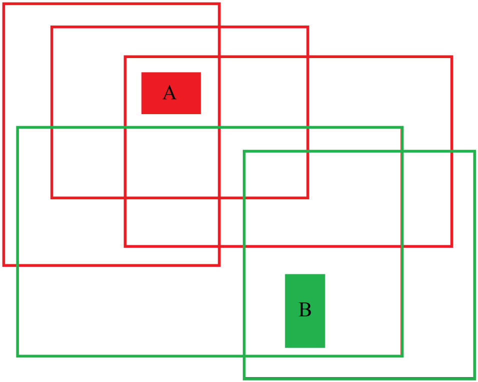

Figure: 2-punching a family of 2-dimensional boxes (A and B have area 1)

Question

parameters	Piercing	Punching
1-dimension, n	✓	?
d-dimenion, 2	✓	?
2-dimension, 3	✓	?

Any h-tuple is k-punchable $\Longrightarrow$ the whole set is k-punchable
Helly-number h?

Results

parameters	Piercing	Punching
1-dimension, n	✓	✓
d-dimenion, 2	✓	✓✗
2-dimension, 3	✓	?

Any h-tuple is k-punchable $\implies$ the whole set is k-punchable
Helly-number h?

Results

- 1-dimensional n-punching
  - **Proposition 1:** $h = n + 1$
- d-dimensional 2-punching
  - lower bound
    - **Proposition 2:** $(4d - 2)$-tuples are not enough.
    - **Corollary 2.1:** $h \geq 4d - 1$
  - upper bound
    - **Conjecture:** $h \leq 4d$

Results

Parameters	Piercing	Punching
1-dimension, n	$n+1$	$n+1$
d-dimension, 2	$3d$, $3d-1$	$4d-1 \leq$

Table: Helly numbers for different settings

Proof of Proposition 1

Minkowski difference

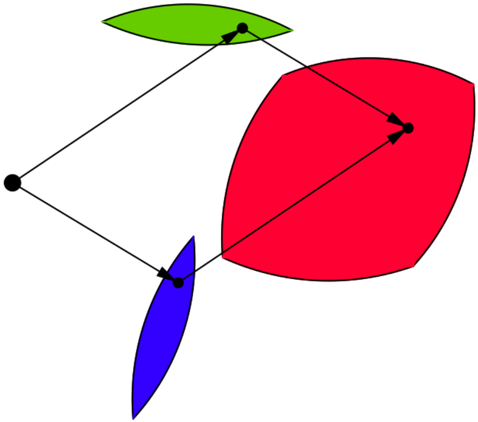

Figure: Minkowski addition, difference

Problem reduces to n-piercing intervals

Proof of Proposition 2

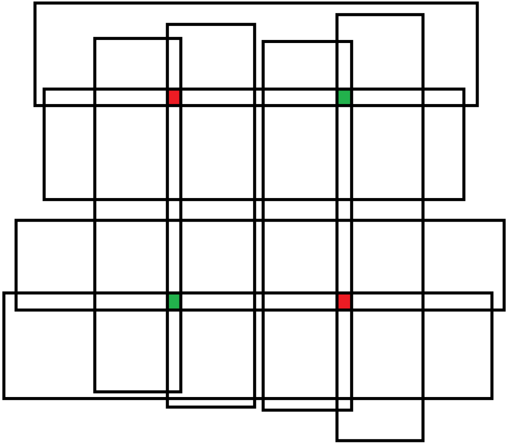

Figure: Construction for $d = 2$. Punching pairs are of the same color.

Proof of Proposition 2

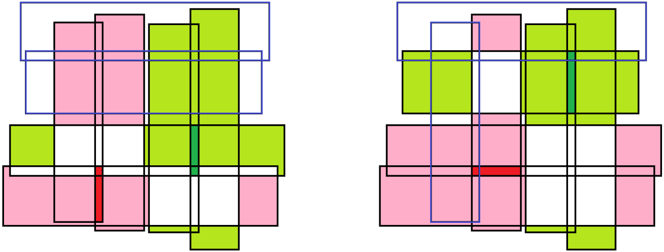

Figure: Any 6-tuple can be punched by 2 big boxes.

Discussion

Upper bound:

- 1-punching: $h = 2d$
- 1 box: $2d$ facets
- The facets of the punching box are determined by a subfamily of size at most $2d$

Conjecture: $h \leq 4d$ for 2-punching

- 2 boxes : $4d$ facets in total

Upper bound problems

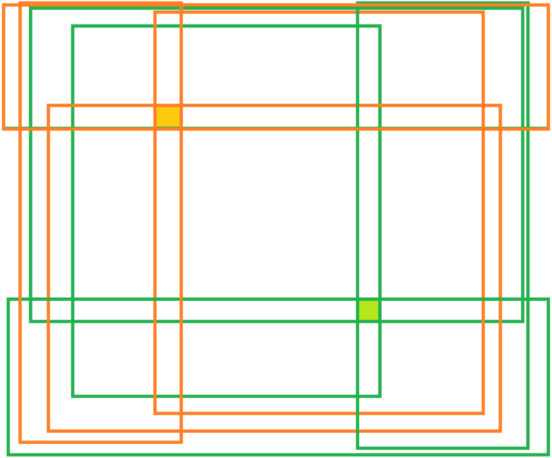

Figure: The highlithted boxes are bordered by the boxes of given color

Upper bound problems

Regrouping the tuples is a problem

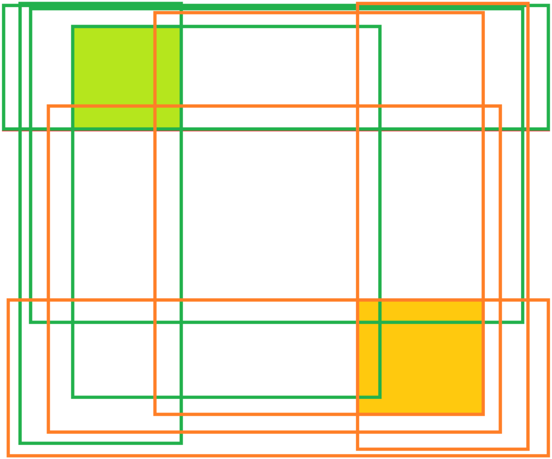

Figure: The highlighted boxes are the maximal punching boxes of this tuple

References

- Damásdi, G., Viktória Földvári, V. & Naszódi,M. (2020). Colorful Helly-type theorems for the volume of intersections of convex bodies. Journal of Combinatorial Theory.
- Chakraborty. S., Ghosh, A. & Nandi, S. (2022). Coloful Helly Theorem for Piercing Boxes with Two Points.