The largest intersecting family of interval along a fixed cyclic permutation without element contained in more than two other elements

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Abstract

A *cyclic permutation* π of the elements of $X = \{x_1, \dots, x_n\}$ is an ordering of the elements along a cycle. It is known that the size of intersecting family of k-elemment($1\leq\ k\ \leq\ \frac{n}{2}$) interval along π at most k. And the size of inclusion-free family of interval along π is at most n. For the reason that an intersecting family A of interval along a fixed cyclic permutation π without element contained in more than two other elements must contain an inclusion-free intersecting family . We can obtain a boundary of the largest size of A by these results. On the basis of the boundary, this article continues to explore the structure of the largest size of A and determine its value.

Keywords: cyclic permutation; inclusion-free family; intersecting family; P-free

1 Introduction and a simple boundary

A *cyclic permutation* π of the elements of $X = \{x_1, \dots, x_n\}$ is an ordering of the elements along a cycle. A set $a\subset X$ is called interval(along π) if its element are consecutive along π. A set $a\subset X$ is called *interval* (along π) if its element are consecutive along π.

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For the poset $P = \{x, y, z\}$ such that x⊂y and x⊂z. An intersecting family A of interval along a fixed cyclic permutation π without element contained in more than two other elements is actually a $P-free$ intersecting family. Let $\bar{A}$ be the largest such family. It is easy to found that when $n \leq 2$, $|\bar{A}| = n$ and when $n = 3$ *or* 4, $|\bar{A}| = n+1$. So in this report we talk about $n \geq 5$. First, It is easy to found such family with size $n+1$. Such as $\{a : all\ the\ intervals\ with\ size\ n-1\} \cup \{a : the\ intervals\ with\ size\ n\}$. So $|\bar{A}| \geq n+1$.

We divide A into two part. Let $M_A \in A$ is the largest inclusion-free intersecting family of A such that for each $b \in A/M_A$, $\exists\ a \in M_A$, $a \subset b$.

From the following lemma we can obtain a simple bound of $|\bar{A}|$.

Lemma 1.1. *[1] If* $\forall\ a \in M_A$, $|a| \leq \frac{n}{2}$, *then* $|M_A| \leq \frac{n}{2}$.

Proof. Denote $|M_A|$=m. Since each two distinct elements ofM_A must have different start position, we can reperesent each element ofM_A by its start position: $M_A = \{{i_1}^*, {i_2}^*, \ldots, {i_m}^*\}$. And let ${i_1}^* = 1^*$ be the shortest interval. Any point of 1^* can serve at most once as a starting point and once as endpoint. For the reason that for any b$\in M_A$,c$\in M_A$,|b|+|c|$\leq n$, we cannot have a member b with endpoint j and a member c with starting point j+1 $(1 \leq |1^*| - 1)$ simutaneously in 1^*. So only one of such b and c can exist. If 1^* is included we obtain $|M_A| \leq |1^*| \leq \frac{n}{2}$.

□

Theorem 1.2. $n + 1 \leq |\bar{A}| \leq 2n$, *and* $\exists\ a \in \bar{A}$, $|a| > \frac{n}{2}$.

Proof. For $\forall\ a \in M_A$, there exists at most one element $b \in A/M_A$ such that $a \subset b$. So $|A/M_A| \leq |M_A|$ and we can know that $|\bar{A}| \leq 2|M_A|$. Form $|\bar{A}| \geq n+1$ and lemma 1.1, we know that there must exist $a \in \bar{A}$,$|a| > \frac{n}{2}$. Since each two distinct elements of M_A must have different start position, we claim that $|M_A| \leq n$.

□

2 Largest size and the way to construct

Theorem 2.1. *When* $|M_A| = n$ $|A| \leq \lfloor \frac{3}{2}n \rfloor$

Proof. Since each two distinct elements ofM_A must have different start position, we can reperesent each element ofM_A by its start position: $M_A = \{1^*, 2^*, \ldots, n^*\}$. We claim that $\forall i^* \in M_A$, we have $|i^*| \leq |(i+1)^*|$ $(i\ mod\ n)$ [Unless the endpoint of $(i+1)^*$ is $i + |(i+1)^*| \leq i - 1 + |i^*|$ (the right side of the inequality is the end point of i^*), we get

$(i+1)^* \subset i^*$.] From $|n^*| \leq |1^*| \leq |2^*| \leq \ldots \leq |n^*|$, we get $|1^*| = |2^*| = \ldots = |n^*| = k$. When $k \leq \frac{n}{2}$ *or* $k = n-1$ *or* $k = n$ the result is trivial. We consider $\frac{n}{2} < k < n-1$.

For $\forall b \in A/M_A$, $\exists i^* \subset b$, then at least one of $(i-1)^*$ *and* $(i+1)^*$ is subset of b, so $|A/M_A| \leq \frac{|M_A|}{2} = \frac{n}{2}$, we get $|A| \leq \lfloor \frac{3}{2}n \rfloor$

□

Observation 1. *For $n \geq 5$, we can always found a a $P-free$ intersecting family A of intervals with $|A| = \lfloor \frac{3}{2}n \rfloor$. For example $A = \{i^* : |i^*| = \lfloor \frac{n}{2} \rfloor + 1,\ i = 1, 2, \ldots, n\} \cup \{i^* : |i^*| = \lfloor \frac{n}{2} \rfloor + 2,\ i$ is even, $i = 1, 2, \ldots, n\}$.*

Theorem 2.2. *When $|M_A| \leq n-1$. If $|M_A| = n-1$ and n is odd, $|A| \leq \lfloor \frac{3}{2}n \rfloor$. Otherwise $|A| < \lfloor \frac{3}{2}n \rfloor$*

Proof. We denote $|M_A| = m < n$, $M_A = \{{i_1}^*, {i_2}^*, \ldots . {i_m}^*\}$, $1 = i_1 < i_2 < \cdots < i_m < n$. When $m < \frac{3}{4}n$, $|A| < \frac{3}{2}n$, the result is trivial. We consider $m \geq \frac{3}{4}n$.

First we discuss the case $i_1 i_2 i_3 \ldots i_{m-1} i_m$ are consecutive and then extend the result to other cases.

If $i_1 i_2 i_3 \ldots i_{m-1} i_m$ are consecutive, $M_A = \{1^*, 2^*, \ldots, m^*\}$, then for $\forall b \in A/M_A$, $t^* \subset b$ but $t - 1^* \nsubseteq b$, $t + 1^* \nsubseteq b$ is possible only if

$$(the\ end\ point\ of\ t+1^*) t + |t+1^*| \geq t - 1 + |t^*| (the\ end\ point\ of\ t^*) \tag{2.1}$$

Let T be the number of such t^* among M_A, then $|A/M_A| \leq \frac{m}{2} + T$.

By inequality 2.1

$$(the\ end\ point\ of\ m^*) m - 1 + |m^*| \geq m - 1 + |1^*| + T \tag{2.2}$$

$$(the\ end\ point\ of\ m^*) m - 1 + |m^*| < n + |1^*| (the\ end\ point\ of\ 1^*) \tag{2.3}$$

$$\Rightarrow T < n - m + 1 \Rightarrow T \leq n - m \tag{2.4}$$

We get $|A/M_A| \leq \frac{m}{2} + T \leq n - \frac{m}{2}$ and $|A| \leq n + \frac{m}{2}$. When $m = n-1$ and n *is odd*, $|A| \leq \lfloor \frac{3}{2}n \rfloor$. Otherwise $|A| < \lfloor \frac{3}{2}n \rfloor$.

We extend the result in the case that $i_1 i_2 i_3 \ldots i_{m-1} i_m$ are not consecutive. Observing the permutation $i_1 i_2 i_3 \ldots i_{m-1} i_m$, we can found that it is union of some segments and $m < n-1$.

Assume the permutation is constructed by d segments: $i_{j1} i_{j2} \ldots i_{jm_j}$ $j = 1, 2, \ldots, d$. Let i_{11}=1 and $i_{dm_d} < n$. For each $j = 1, 2, \ldots, d$, let $|i_{j1}| = k_j$, T_j be the number of elements t^* among each $B_j = \{{i_{j1}}^*, {i_{j2}}^*, \ldots, {i_{jm_j}}^*\}$ has property 2.1. We get $m = m_1 + m_2 + \cdots + m_d$, $T = T_1 + T_2 + \cdots + T_d$ and inequality

$$i_{j1} - 1 + k_j + m_j - 1 < (the\ end\ position\ of\ {i_{(j+1)1}}^*) \tag{2.5}$$

$$\Rightarrow i_{j1} - 1 + k_j + m_j - 1 \leq i_{(j+1)1} - 2 + k_{j+1} \tag{2.6}$$

From calculation

$$\left.\begin{array}{l} t_1 + i_{11} - 1 + k_1 + m_1 - 1 \leq i_{21} - 2 + k_2 \\ t_1 + i_{11} - 1 + k_1 + m_1 - 1 \leq i_{21} - 2 + k_2 \\ t_2 + i_{21} - 1 + k_2 + m_2 - 1 \leq i_{21} - 2 + k_3 \\ \qquad\qquad\qquad\qquad \cdots \\ t_d + i_{d1} - 1 + k_d + m_d - 1 \leq i_{11} - 2 + k_1 \end{array}\right. \Rightarrow t + m \leq n \Rightarrow t \leq n - m$$

We get $|A/M_A| \leq \frac{m}{2} + T \leq n - \frac{m}{2}$ and $|A| \leq n + \frac{m}{2}$. For $m < n - 1$, we have $|A| < \lfloor \frac{3}{2} n \rfloor$ □

Corollary 2.3. $|\bar{A}| = \lfloor \frac{3}{2} n \rfloor$.

Theorem 2.4. *$\forall a \in \bar{A}$, $|a| \geq \lfloor \frac{n}{2} \rfloor + 1$.*

Proof. we know $|\bar{A}| = \lfloor \frac{3}{2} n \rfloor$ and it can happen only if $|M_A| = n - 1, n \text{ } is \text{ } odd$ or $|M_A| = n$. From the proof of theorem 2.1, when $|M_A| = n$, the result is trivial.

For $|M_A| = n - 1, n \text{ } is \text{ } odd$, we denote $M_A = \{1^*, 2^*, \ldots, n - 1^*\}$ and $|1^*| < |2^*| < \cdots < |n - 1^*|$, let $|1^*| = k$, we get

$$\begin{cases} n - 1 + |n - 1^*| \leq n + k - 1 \\ if \ k \leq \frac{n}{2}, \ then \ k + 1 - 1 + |k + 1^*| \geq n - k + 1 \end{cases} \Rightarrow \begin{cases} |n - 1^*| \leq k + 1 \\ |k + 1^*| \geq n - k + 1 \end{cases}$$

$\Rightarrow n - k + 1 \leq k + 1 \Rightarrow 2k \geq n \Rightarrow k \geq \frac{n}{2}$

When n is odd, there is a contradiction.

□

References

[1] Gerbner.D., Patkos.B. (2018). Extremal Finite Set Theory. 10.1201/9780429440809.