Communication complexity problem

Advisor: Hegyvári Norbert

Kovács Fruzsina

2023/2024/I.

Background of the problem: Communication complexity

- **Communication complexity**
  The quantity of a form of communication to solve a certain problem, with an input between more than one parties (introduced by **A. Yao 1959**). The communication problem usually runs around more participants, to whom each belongs an n-bit long information.

Background of the problem: Communication complexity

- **Communication complexity**
  The quantity of a form of communication to solve a certain problem, with an input between more than one parties (introduced by **A. Yao 1959**). The communication problem usually runs around more participants, to whom each belongs an n-bit long information.

- **Log-rank conjecture**
  One of the main conjecture in connection with complexity, which states a polinomial connection between the function of a two-party communication complexity problem and the rank of its input matrix.

Background of the problem: Communication complexity

- **Communication complexity**
  The quantity of a form of communication to solve a certain problem, with an input between more than one parties (introduced by **A. Yao 1959**). The communication problem usually runs around more participants, to whom each belongs an n-bit long information.

- **Log-rank conjecture**
  One of the main conjecture in connection with complexity, which states a polinomial connection between the function of a two-party communication complexity problem and the rank of its input matrix.

- There is a close link between theoretical computer science and arithmetic, additive combinatorics can act as a useful tool.

Communication complexity

Definition (Complexity)

In case of a two-party communication problem:

$$S(f) := \min_{\text{all protocols}} \max_{x,y}\{\text{numbers of bits exchanged by the protocol on } x; y\}$$

- The definition of complexity is similar even if the number of participants is more than two.

Definition of the problem

- Three-party communication problem with players A, B and C with the inputs x, y and z

Definition of the problem

- Three-party communication problem with players A, B and C with the inputs x, y and z
- Number-on-forehead model:

  A knows: x, y
  B knows: y, z
  C knows: x, z

Definition of the problem

- Three-party communication problem with players A, B and C with the inputs x, y and z

- Number-on-forehead model:

  A knows: x, y
  B knows: y, z
  C knows: x, z

- $F : [n]^3 \rightarrow \{0; 1\}$ where $F(x, y, z) = 1$ if $x + y + z = n$ and 0 otherwise for a given $n \in \mathbb{N}^+$

Definition of the problem

- Three-party communication problem with players A, B and C with the inputs x, y and z
- Number-on-forehead model:

  A knows: x, y
  B knows: y, z
  C knows: x, z

- $F : [n]^3 \rightarrow \{0; 1\}$ where $F(x, y, z) = 1$ if $x + y + z = n$ and 0 otherwise for a given $n \in \mathbb{N}^+$
- Algorithm to solve the $n = x + y + z$ equation with the least possible complexity

Definition of the problem

- Three-party communication problem with players A, B and C with the inputs x, y and z

- Number-on-forehead model:

  A knows: x, y
  B knows: y, z
  C knows: x, z

- $F : [n]^3 \to \{0;1\}$ where $F(x,y,z) = 1$ if $x + y + z = n$ and 0 otherwise for a given $n \in \mathbb{N}^+$

- Algorithm to solve the $n = x + y + z$ equation with the least possible complexity

- Complexity of the trivial solution: the length of x (in bits): $\log_2 x < \log_2(n-2)$

Definition of the problem

- Three-party communication problem with players A, B and C with the inputs x, y and z
- Number-on-forehead model:

  A knows: x, y
  B knows: y, z
  C knows: x, z

- $F : [n]^3 \rightarrow \{0; 1\}$ where $F(x, y, z) = 1$ if $x + y + z = n$ and 0 otherwise for a given $n \in \mathbb{N}^+$
- Algorithm to solve the $n = x + y + z$ equation with the least possible complexity
- Complexity of the trivial solution: the length of x (in bits): $\log_2 x < \log_2(n - 2)$
- Complexity of a random process: logarithmic in the input size, at most a loglog(n) communication.

Deterministic Protocol

- Method of coloring integers from 1 to n, as colors can be calculated by the players, but the complexity decreases

Deterministic Protocol

- Method of coloring integers from 1 to n, as colors can be calculated by the players, but the complexity decreases
- k: the number of the used color class

Deterministic Protocol

- Method of coloring integers from 1 to n, as colors can be calculated by the players, but the complexity decreases
- k: the number of the used color class
- 3-AP-free: each color class is free from a three-term arithmetic progression

Deterministic Protocol

- Method of coloring integers from 1 to n, as colors can be calculated by the players, but the complexity decreases
- k: the number of the used color class
- 3-AP-free: each color class is free from a three-term arithmetic progression
- difference in the AP:

$$d := n - (x + y + z)$$

Deterministic Protocol

- Method of coloring integers from 1 to n, as colors can be calculated by the players, but the complexity decreases
- k: the number of the used color class
- 3-AP-free: each color class is free from a three-term arithmetic progression
- difference in the AP:
$$d := n - (x + y + z)$$
- Calculations:

A: $2x + y$

C: $2x + y + d = 2x + y + n - x - y - z = n + x - z$

B: $2x + y + 2d = (n + x - z) + d = x - z + n + n - x - y = 2n - 2z - y$

$d = 0 \Leftrightarrow$ all of the three sent colors are the same

Sketch of the coloring

Theorem (Behrend)

One can color the set $[n]$ with $2^{O(\sqrt{\log n})}$ colors with no monochromatic three-term arithmetic progression.

- We can overestimate the complexity from the number of the possible 3-AP-free colorings

Sketch of the coloring

Theorem (Behrend)

One can color the set $[n]$ *with* $2^{O(\sqrt{\log n})}$ *colors with no monochromatic three-term arithmetic progression.*

- We can overestimate the complexity from the number of the possible 3-AP-free colorings
- Two parameters d, r and writing each number in base $(2d+1)$

Sketch of the coloring

Theorem (Behrend)

One can color the set $[n]$ *with* $2^{O(\sqrt{\log n})}$ *colors with no monochromatic three-term arithmetic progression.*

- We can overestimate the complexity from the number of the possible 3-AP-free colorings
- Two parameters d, r and writing each number in base $(2d+1)$
- For every $x \in [n]$

$$x = x_0 + x_1(2d+1) + \ldots x_j(2d+1)^j + \ldots x_r(2d+1)^r,$$

where $x_j \in [d], j = 1, 2, \ldots r$

Sketch of the coloring

Theorem (Behrend)

One can color the set $[n]$ *with* $2^{O(\sqrt{\log n})}$ *colors with no monochromatic three-term arithmetic progression.*

- We can overestimate the complexity from the number of the possible 3-AP-free colorings
- Two parameters d, r and writing each number in base $(2d + 1)$
- For every $x \in [n]$

  $$x = x_0 + x_1(2d + 1) + \ldots x_j(2d + 1)^j + \ldots x_r(2d + 1)^r,$$

  where $x_j \in [d], j = 1, 2, \ldots r$
- For each x there is a one-to-one map to v, where $v = (x_0, x_1, \ldots, x_r) \in \mathbb{N}^r$.

Sketch of the coloring

Theorem (Behrend)

One can color the set $[n]$ *with* $2^{O(\sqrt{\log n})}$ *colors with no monochromatic three-term arithmetic progression.*

- We can overestimate the complexity from the number of the possible 3-AP-free colorings
- Two parameters d, r and writing each number in base $(2d+1)$
- For every $x \in [n]$

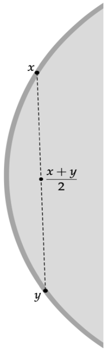
  $$x = x_0 + x_1(2d+1) + \ldots x_j(2d+1)^j + \ldots x_r(2d+1)^r,$$

  where $x_j \in [d], j = 1, 2, \ldots r$
- For each x there is a one-to-one map to v, where $v = (x_0, x_1, \ldots, x_r) \in \mathbb{N}^r$.
- A color class corresponds to the surface of a sphere, where the norm $\|v\|$ is constant:

  $$\|v\| = \sqrt{\sum_{i=1}^{n} x_i^2}$$

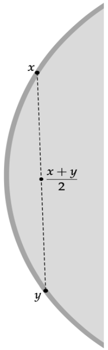
Sketch of the coloring

- Given two points on the sphere, they can form an arithmetic series only with the third point in their bisector → 3-AP-free condition fulfills



Sketch of the coloring

- Given two points on the sphere, they can form an arithmetic series only with the third point in their bisector → 3-AP-free condition fulfills



- Optimizing the parameters as $r = \sqrt{\log n}$ and $d = 2^{O(\sqrt{\log n})}$ we can obtain that the total number of the colorings equals to $2^{O(\sqrt{\log n})}$

Complexity of the problem

- Two of the participants (A and B) send their colors for the third party and C will send back the number 1, if the equation fulfills, and a number 0 otherwise
  $\Rightarrow$ the complexity of this protocol is $2\log_2 k + 2$

Complexity of the problem

- Two of the participants (A and B) send their colors for the third party and C will send back the number 1, if the equation fulfills, and a number 0 otherwise
  $\Rightarrow$ the complexity of this protocol is $2\log_2 k + 2$

- Based on the adaptation to the sphere, the total number of the colorings is $2^{O(\sqrt{\log n})}$

Complexity of the problem

- Two of the participants (A and B) send their colors for the third party and C will send back the number 1, if the equation fulfills, and a number 0 otherwise
  $\Rightarrow$ the complexity of this protocol is $2\log_2 k + 2$

- Based on the adaptation to the sphere, the total number of the colorings is $2^{O(\sqrt{\log n})}$

- From estimating k with the formula we can get the complexity of the problem:

Theorem

$S(F) = O(\sqrt{\log n})$.

- This estimation is the best known.