Név: **Gálffy Veronika**
NEPTUN-kód: **OHUR40**
Témavezető: **Dr. Kovács Sándor**

Math project 3
Stability and bifurcations in reaction-diffusion equations modeling the spread of infectious diseases

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1 Biological feasibility

In [8] the authors studied the following ODE system, which models the spread of an infection disease in a human population:

$$(\dot{S}, \dot{E}, \dot{I}) = \mathbf{f}(S, E, I) \tag{1}$$

where

$$\left.\begin{array}{rcl} f_1(S,E,I) & := & \lambda - \dfrac{aSI}{S+I} + \beta I - \psi S - \delta_S S, \\ f_2(S,E,I) & := & \psi S + \kappa I - \delta_E E, \\ f_3(S,E,I) & := & \dfrac{aSI}{S+I} - \kappa I - \beta I - \delta_I I. \end{array}\right\} \tag{2}$$

Suppose that for the modeling of the spread of the disease we also need to consider the spatial spread of the population: members of the population can move inside the domain Ω with (piecewise) smooth boundary, and there is no migration through the boundary of the domain. The system can be described by the reaction-diffusion differential equation

$$\partial_t \mathbf{u} = D\Delta_{\mathbf{r}}\mathbf{u} + \mathbf{f}(\mathbf{u}) \tag{3}$$

with homogeneous Neumann-boundary conditions

$$(\mathbf{n}\cdot\nabla_{\mathbf{r}})\mathbf{u}(\mathbf{r},t) = \mathbf{0} \qquad ((\mathbf{r},t)\in\partial\Omega\times\mathbb{R}_0^+), \tag{4}$$

and with (non identical vanishing) non-negative initial condition

$$\mathbf{u}(\cdot,t) = \mathbf{u}_0(\cdot) \qquad ((\mathbf{r},t)\in\overline{\Omega}\times\{0\}), \tag{5}$$

where

$$D := \begin{bmatrix} d_S & 0 & 0 \\ 0 & d_E & 0 \\ 0 & 0 & d_I \end{bmatrix}$$

is the positive definite diffusion matrix, furthermore

$$\mathbf{u} := (S, E, I) \qquad \text{and} \qquad \mathbf{f} := (f_1, f_2, f_3).$$

Last semester, we showed that the model is biologically feasible in the sense that the positive quadrant is an invariant region, that is solutions with positive initial values stay positive. Now we aim to prove the boundedness of solutions under some conditions.

Theorem. Suppose that D is a scalar matrix, i.e. $d_S = d_E = d_I$ holds. Then model (3) defined on $\Omega \times \mathbb{R}_0^+$ has bounded solution for any initial conditions satisfying (5).

Proof. Let us define

$$\sigma(\mathbf{r}, t) :\equiv \sigma(S, E, I) = S(\mathbf{r}, t) + E(\mathbf{r}, t) + I(\mathbf{r}, t),$$

then from $d_S = d_E = d_I = d$, and summing the equations, we get for the time derivative of σ the inequality

$$\dot{\sigma}(\mathbf{r}, t) - d\Delta\sigma(\mathbf{r}, t) \equiv \lambda - \delta_S S - \delta_E E - \delta_I I \leq \lambda - \xi\sigma(\mathbf{r}, t), \tag{6}$$

where $\xi = \min\{d_S, d_E, d_I\}$. Thus $\sigma(\mathbf{r}, t)$ satisfies

$$\dot{\sigma}(\mathbf{r}, t) - d\Delta\sigma(\mathbf{r}, t) \leq \lambda - \xi\sigma(\mathbf{r}, t), \qquad \sigma(\mathbf{r}, 0) \geq 0. \tag{7}$$

Let ϕ be the solution of the initial value problem

$$\dot{y}(t) \equiv \lambda - \xi y(t) \quad (t \in [0, +\infty)), \qquad y(0) = \max_{\mathbf{r}\in\overline{\Omega}} \sigma(\mathbf{r}, 0). \tag{8}$$

Thus, ϕ is bounded:

$$\phi(t) \leq \max\left\{\lambda/\xi, \max_{\mathbf{r}\in\overline{\Omega}} \sigma(\mathbf{r}, 0)\right\}$$

and by the Comparison theorem (cf. [1]), $\sigma(\mathbf{r}, t) \leq \phi(t)$ for every $t \geq 0$, and hence

$$\overline{\Omega} \times \mathbb{R}_0^+ \ni (\mathbf{r}, t) \mapsto S(\mathbf{r}, t), E(\mathbf{r}, t), I(\mathbf{r}, t)$$

are bounded, too. We remark only that (8) implies that $\lim_{t\to+\infty} \phi(t) = \lambda/\xi$, thus $\sigma(\mathbf{r}, \cdot)$ $(\mathbf{r} \in \overline{\Omega})$ is defined on the whole positive half line and

$$\limsup_{t\to+\infty} \max_{\mathbf{r}\in\overline{\Omega}} \sigma(\mathbf{r}, t) \leq \lambda/\xi. \quad \blacksquare$$

Let us consider the case when D is not a scalar matrix, i.e. the diffusion coefficient are unequal and

$$\psi + \delta_S = \kappa + \delta_I =: \mu \tag{9}$$

holds.

Theorem. Assumption (9) implies that system (3) is bounded.

Proof.

Step 1. Let us define the total populations size at time t by

$$N(t) := \frac{1}{|\Omega|} \int_\Omega (S(\mathbf{r}, t) + I(\mathbf{r}, t))\, d\mathbf{r} \qquad (t \in [0, +\infty))$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Thus,

$$N_0 := N(0) = \frac{1}{|\Omega|} \int_\Omega (S(\mathbf{r}, 0) + I(\mathbf{r}, 0))\, d\mathbf{r}.$$

Adding the first and the third equation in (3) and integrating over Ω, we have

$$\frac{d}{dt}\int_\Omega (S(\mathbf{r},t)+I(\mathbf{r},t))\ d\mathbf{r} = \lambda|\Omega| - \mu\int_\Omega (S+I) + d_{SS}\int_\Omega \Delta S + d_{II}\int_\Omega \Delta I.$$

Using the divergence theorem we get for $\phi \in \{S, I\}$

$$\int_\Omega \Delta\phi = \int_\Omega \nabla(\nabla\phi) = \int_{\partial\Omega} \nabla\phi\cdot\mathbf{n}\, d\Sigma = 0,$$

because homogeneous Neumann boundary conditions means in (4) that

$$\nabla_{\mathbf{r}}S = 0, \quad \nabla_{\mathbf{r}}E = 0, \quad \nabla_{\mathbf{r}}I = 0 \qquad \text{on} \qquad \partial\Omega\times\mathbb{R}_0^+.$$

Hence we have

$$\frac{d}{dt}N(t) + \mu N(t) = \lambda \qquad (t\in[0,+\infty)).$$

Integrating this equality we have

$$N(t) = \left(N_0 - \frac{\lambda}{\mu}\right)e^{-\mu t} + \frac{\lambda}{\mu} \qquad (t\in[0,+\infty)).$$

This means that for $\phi\in\{S,I\}$ we have

$$\int_\Omega \phi \le \frac{\lambda}{\mu}.$$

Hence ϕ is bounded by $\frac{\lambda}{\mu}$ a.e. in Ω, i.e.

$$\|\phi\|_\infty \le \frac{\lambda}{\mu} \qquad (\phi\in\{S,I\}).$$

Step 2. Using the method presented in [4] we can see that (9) implies that the second variable in (3), resp. in (2) is also bounded:

$$E \le \max\left\{\frac{\mu N_0}{\delta_E},\ \max_{\mathbf{r}\in\overline{\Omega}} E(\mathbf{r},0),\ \frac{a\kappa\lambda + \lambda(\beta+\delta_I+\kappa)(\psi-\kappa)}{\delta_E\{a(\delta_I+\kappa)+(\beta+\delta_I+\kappa)(\delta_S-\delta_I-\kappa+\psi)\}}\right\}. \quad \blacksquare$$

In [8] the authors showed, that system (2) has two equilibria, where if the reproduction ratio is denoted by

$$\mathcal{R}_0 := \frac{a}{\kappa+\beta+\delta_I},$$

then

- in case $\mathcal{R}_0 < 1$ system (2) has one equilibrium:

$$\mathfrak{E}_b := \left(\frac{\lambda}{\delta_S+\psi}, \frac{\lambda\psi}{\delta_E(\delta_S+\psi)}, 0\right),$$

- in case $\mathcal{R}_0 > 1$ system (2) has two equilibira, $\mathfrak{E}_b$ and $\mathfrak{E}_e := (S_e, E_e, I_e)$, where

$$\begin{aligned} S_e &:= \frac{\lambda(\beta+\delta_I+\kappa)}{a(\delta_I+\kappa)+(\beta+\delta_I+\kappa)(\psi+\delta_S-\delta_I-\kappa)}, \\ E_e &:= \frac{a\kappa\lambda+\lambda(\beta+\delta_I+\kappa)(\psi-\kappa)}{\delta_E\{a(\delta_I+\kappa)+(\beta+\delta_I+\kappa)(\delta_S-\delta_I-\kappa+\psi)\}}, \\ I_e &:= \frac{\lambda(a-\beta-\delta_I-\kappa)}{a(\delta_I+\kappa)+(\beta+\delta_I+\kappa)(\delta_S-\delta_I-\kappa+\psi)}. \end{aligned}$$

These equilibria – as constant solutions – are solutions of system (3), too. The stability of these solutions can be determined by the method of linearization.

2 Linear stability

A spatially constant solution $\mathbf{\Phi}(\cdot) = (\Phi_1(\cdot), \Phi_2(\cdot))$ of system (3) satisfies obvious boundary conditions (4) and system (2). The equilibria $\mathfrak{E}_b$ and $\mathfrak{E}_e$ of system (2) are constant solutions of (3), (4) at the same time. In order to investigate the local stability of these constant solutions of (3) we linearize (3) at $\mathfrak{E}_b$ and $\mathfrak{E}_e$. The linearized system with the same initial and boundary conditions has the form

$$\left.\begin{array}{ll} \partial_t \mathbf{v} = D \cdot \Delta_{\mathbf{r}} \mathbf{v} + \mathfrak{A}\mathbf{v} & \text{in } \Omega \times \mathbb{R}_0^+, \\ (\mathbf{n} \cdot \nabla_{\mathbf{r}})\, \mathbf{v}(\mathbf{r}, t) = \mathbf{0} & (\mathbf{r}, t) \in \partial\Omega \times \mathbb{R}_0^+, \\ \mathbf{v}(\mathbf{r}, 0) = \mathbf{v}_0(\mathbf{r}) & (\mathbf{r}, t) \in \overline{\Omega} \times \{0\} \end{array}\right\} \tag{10}$$

where

$$\mathfrak{A} := J_{(f_1, f_2, f_3)}(\overline{\mathbf{u}}) =: \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad /\overline{\mathbf{u}} \in \{\mathfrak{E}_b,\ \mathfrak{E}_e\}/\,.$$

Using Fourier method we suppose that system (10) has solutions of the form

$$\mathbf{\Lambda}(\mathbf{r}, t) = \psi(\mathbf{r}) \cdot \boldsymbol{\varphi}(t) \qquad (\mathbf{r}, t) \in \overline{\Omega} \times \mathbb{R}_0^+)$$

where

$$\psi(\mathbf{r}) : \overline{\Omega} \to \mathbb{R}, \qquad \text{resp.} \qquad \boldsymbol{\varphi} : \mathbb{R}_0^+ \to \mathbb{R}^3$$

satisfy

$$\dot{\boldsymbol{\varphi}} = (\mathfrak{A} - \lambda D)\boldsymbol{\varphi} \tag{11}$$

and

$$\Delta\psi = -\lambda\psi, \qquad \left.\frac{\partial\psi}{\partial\mathbf{n}}\right|_{\partial\Omega} = 0. \tag{12}$$

Thus, for the spatial domain Ω the solutions of problem (10) have the form

$$\mathbf{\Lambda}(\mathbf{r}, t) = \sum_{n=0}^{\infty} \psi_n(\mathbf{r}) \exp(\mathfrak{A}_n t)\, \mathbf{\Psi}_{0_n} \qquad ((\mathbf{r}, t) \in \overline{\Omega} \times \mathbb{R}_0^+)$$

(cf. [7]), where for $n \in \mathbb{N}_0$

$$\mathfrak{A}_n := \mathfrak{A} - \lambda_n D, \qquad \mathbf{\Lambda}_{0_n} := \int_{\Omega} \mathbf{v}_0(\mathbf{r})\, \psi_n(\mathbf{r})\, d\mathbf{r}$$

and λ_n is the n-th eigenvalue of the minus Laplacian on Ω subject to homogeneous Neumann boundary conditions i.e. the solution of (12), resp. ψ_n is the corresponding normalized eigenfunction, i.e. ψ_n solves (12). It is well known (cf. [6]) that

$$0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \longrightarrow +\infty \qquad (n \to \infty)$$

and the eigenfunctions to different eigenvalues are orthogonal to each other.

According to [1], [2] the equilibrium $\overline{\mathbf{u}}$ of (3) is asymptotically stable if for all $n \in \mathbb{N}_0$ the matrix $\mathfrak{A}_n$ is Hurwitz stable, i.e. all eigenvalues of $\mathfrak{A}_n$ have negative real part; furthermore $\overline{\mathbf{u}}$ is unstable if for some index $n \in \mathbb{N}_0$ there exists an eigenvalue of $\mathfrak{A}_n$ with positive real part. The characteristic polynomial of the matrix $\mathfrak{A}_n$ has the form

$$\Delta_{\mathfrak{A}_n}(z) := z^3 - \mathfrak{T}_n z^2 + \widetilde{\mathfrak{A}}_n z - \mathfrak{D}_n \qquad (z \in \mathbb{C}) \tag{13}$$

where

$$\mathfrak{D}_n := \det(\mathfrak{A}_n) = \det \begin{bmatrix} a_{11} - d_{SS} & a_{12} & a_{13} \\ a_{21} & a_{22} - d_{EE} & a_{23} \\ a_{31} & a_{32} & a_{33} - d_{II} \end{bmatrix}$$

and

$$\mathfrak{T}_n := \operatorname{Tr}(\mathfrak{A}_n) = \operatorname{Tr}(\mathfrak{A}) - \lambda_n \operatorname{Tr}(D) = a_{11} + a_{22} + a_{33} - \lambda_n \left(d_{SS} + d_{EE} + d_{II}\right),$$

resp. $\widetilde{\mathfrak{A}}_n$ is the sum of 2-by-2 determinants obtained from the matrix $\mathfrak{A}_n$ after omitting the rows and columns with the same index:

$$\widetilde{\mathfrak{A}}_n := \det \begin{bmatrix} a_{11} - d_{SS} & a_{12} \\ a_{21} & a_{22} - d_{EE} \end{bmatrix} + \det \begin{bmatrix} a_{11} - d_{SS} & a_{13} \\ a_{31} & a_{II} - d_{II} \end{bmatrix} + \det \begin{bmatrix} a_{22} - d_{EE} & a_{23} \\ a_{32} & a_{33} - d_{II} \end{bmatrix},$$

i.e. $\widetilde{\mathfrak{A}}_n$ is the sum of all 2×2 principal minors of $\mathfrak{A}_n$. The Routh-Hurwitz condition states that $\mathfrak{A}_n$ is Hurwitz stable if and only if for all $n \in \mathbb{N}_0$

$$\mathfrak{T}_n < 0, \qquad \mathfrak{D}_n < 0, \qquad \text{resp.} \qquad \mathfrak{T}_n \cdot \widetilde{\mathfrak{A}}_n < \mathfrak{D}_n \tag{14}$$

hold. In order to show diffusion driven or Turing instability one need to check that the given equilibrium is (locally) asymptotically stable steady state of the kinetic system (2) and it is unstable with respect to (3), i.e. one of the conditions in (14) is violated. The stability with respect to the kinetic system (2) ensures that $\operatorname{Tr}(\mathfrak{A}) < 0$ holds, which has the consequence that $\mathfrak{T}_n < 0$, i.e. (14) reduces to

$$\mathfrak{T}_n \cdot \widetilde{\mathfrak{A}}_n < \mathfrak{D}_n < 0. \tag{15}$$

This means that we have diffusion driven instability if for some $n \in \mathbb{N}$

$$\mathfrak{D}_n > 0 \qquad \text{or} \qquad \mathfrak{T}_n \cdot \widetilde{\mathfrak{A}}_n > \mathfrak{D}_n$$

holds. In case of

- $\bar{\mathbf{u}} = \mathfrak{E}_b$ the characteristic polynomial (13) has the form

$$\Delta^{\mathfrak{E}_b}_{\mathfrak{A}_n}(z) := z^3 + A^{\mathfrak{E}_b} z^2 + B^{\mathfrak{E}_b} z + C^{\mathfrak{E}_b} \qquad (z \in \mathbb{C}) \tag{16}$$

  where

$$\begin{aligned} A^{\mathfrak{E}_b} &:= -a + \beta + \delta_E + \delta_I + \delta_S + \kappa + \left(d_{SS} + d_{EE} + d_{II}\right) \cdot \lambda_n + \psi, \\ B^{\mathfrak{E}_b} &:= -\left(-d_{SS}\lambda_n - \delta_S - \psi\right) \cdot \left(-a + \beta + d_{EE}\lambda_n + d_{II}\lambda_n + \delta_E + \delta_I + \kappa\right) \\ &\quad + \left(-d_{EE}\lambda_n - \delta_E\right) \cdot \left(a - \beta - d_{II}\lambda_n - \delta_I - \kappa\right), \\ C^{\mathfrak{E}_b} &:= \left(d_{EE}\lambda_n + \delta_E\right) \cdot \left(d_{SS}\lambda_n + \delta_S + \psi\right) \cdot \left(-a + \beta + d_{II}\lambda_n + \delta_I + \kappa\right). \end{aligned}$$

- $\bar{\mathbf{u}} = \mathfrak{E}_e$ the characteristic polynomial (13) has the form

$$\Delta^{\mathfrak{E}_e}_{\mathfrak{A}_n}(z) := z^3 + A^{\mathfrak{E}_e} z^2 + B^{\mathfrak{E}_e} z + C^{\mathfrak{E}_e} \qquad (z \in \mathbb{C}) \tag{17}$$

where

$$
\begin{aligned}
A^{\mathfrak{E}_e} &:= a-\beta-\delta_I+\delta_S-\kappa+(d_{SS}+d_{EE}+d_{II})\lambda_n+\psi,\\
B^{\mathfrak{E}_e} &:= -\frac{(-a+\beta+\delta_I+\kappa)^2\left(a\beta-(\beta+\delta_I+\kappa)^2\right)}{a^2}\\
&\quad -(-\delta_E-d_{EE}\lambda_n)(a-\beta-\delta_I+\delta_S-\kappa+d_{SS}\lambda_n+d_{II}\lambda_n+\psi)\\
&\quad -\left(d_{SS}\lambda_n+\frac{a^2+a(-2(\beta+\delta_I+\kappa)+\delta_S+\psi)+(\beta+\delta_I+\kappa)^2}{a}\right)\\
&\quad \cdot\left(\frac{(\beta+\delta_I+\kappa)(-a+\beta+\delta_I+\kappa)}{a}-d_{II}\lambda_n\right),\\
C^{\mathfrak{E}_e} &:= (d_{EE}\lambda_n+\delta_E)\cdot\left\{\left(d_{II}\lambda_n-\frac{(\beta+\delta_I+\kappa)(-a+\beta+\delta_I+\kappa)}{a}\right)\right.\\
&\quad \cdot\left(d_{SS}\lambda_n+\frac{a^2+a(-2(\beta+\delta_I+\kappa)+\delta_S+\psi)+(\beta+\delta_I+\kappa)^2}{a}\right)\\
&\quad \left.-\frac{(-a+\beta+\delta_I+\kappa)^2\left(a\beta-(\beta+\delta_I+\kappa)^2\right)}{a^2}\right\}.
\end{aligned}
$$

In order to examine the real part of the zeros of the characteristic polynomial (13) we are going to use the well-known Cardano formula on the cubic polynomial

$$
p(z) := z^3+a_2z^2+a_1z+a_0 \qquad (z\in\mathbb{C}). \tag{18}
$$

Lemma (Cardano). The zeros of be calculated as follows:

$$
\xi_1 := -\frac{a_2}{3}+u+v, \qquad \xi_2 := -\frac{a_2}{3}-\frac{u+v}{2}+\frac{\imath\sqrt{3}}{2}(u-v), \qquad \xi_3 := -\frac{a_2}{3}-\frac{u+v}{2}-\frac{\imath\sqrt{3}}{2}(u-v),
$$

where

$$
u := \sqrt[3]{\beta+\sqrt{\alpha^3+\beta^2}}, \quad v := \sqrt[3]{\beta-\sqrt{\alpha^3+\beta^2}} \qquad \text{with} \qquad \alpha := \frac{3a_1-a_2^2}{9}, \quad \beta := \frac{9a_1a_2-2a_2^3-27a_0}{54}.
$$

Furthermore, calculating the discriminant

$$
\triangle := \alpha^3+\beta^2
$$

one concludes by means of the sign of $\triangle$ that

- if $\triangle > 0$ then p has one real root: ξ_1 and two complex conjugate roots: ξ_1, $\xi_2 = \overline{\xi_1}$;
- if $\triangle = 0$ then p has only real roots and $\xi_2 = \xi_3$;
- if $\triangle < 0$ then p has three unequal real roots: ξ_1, ξ_2, ξ_3.

Thus, in case of

- $\bar{\mathbf{u}} = \mathfrak{E}_b$ direct calculation shows that

$$\begin{aligned}\triangle &= \alpha^3 + \beta^2 = \left(\frac{3a_1 - a_2^2}{9}\right)^3 + \left(\frac{9a_1a_2 - 2a_2^3 - 27a_0}{54}\right)^2 \\ &= -\frac{1}{108}(-a + \beta - d_{EE}\lambda_n + d_{II}\lambda_n - \delta_E + \delta_I + \kappa)^2 \cdot (d_{SS}\lambda_n - d_{II}\lambda_n - \delta_E + \delta_S + \psi)^2 \\ &\quad \cdot(a - \beta + d_{SS}\lambda_n - d3\lambda_n - \delta_I + \delta_S - \kappa + \psi)^2 < 0.\end{aligned}$$

Hence the roots of (13) are

$$\begin{aligned}\xi_1 &:= -d_{EE}\lambda_n - \delta_E < 0, \\ \xi_2 &:= -d_{SS}\lambda_n - \delta_S - \psi < 0, \\ \xi_3 &:= a - \beta - d_{II}\lambda_n - \delta_I - \kappa = (a - \beta - \delta_I - \kappa) - d_{II}\lambda_n.\end{aligned}$$

Thus, in case of

1. $\mathcal{R}_0 < 1$, i.e. when
$$a - \beta - \delta_I - \kappa < 0$$
holds, then the polynomial in (13) is stable, which means that the boundary equilibrium of the kinetic system (2) remains stable with respect to (3), i.e. no diffusion driven instability occurs.
2. $\mathcal{R}_0 > 1$, i.e. when
$$a - \beta - \delta_I - \kappa > 0 \tag{19}$$
holds, then the polynomial in (13) is unstable, which means the originally unstable boundary equilibrium of the kinetic system may or may not be stable, it is stable if and only if
$$d_{II} > \frac{a - (\beta + \delta_I + \kappa)}{\lambda_1} > \frac{a - (\beta + \delta_I + \kappa)}{\lambda_n} \quad (n \in \mathbb{N}) \tag{20}$$
holds, i.e. the diffusion coefficient of the infecteds crosses a critical value. This is the case of the well known phenomenon when diffusion causes stability. This reflects the fact that if the individuals in the infected compartment are moving toward each other then the uninfected equilibrium stabilizes.

- $\bar{\mathbf{u}} = \mathfrak{E}_e$ direct calculation shows that

$$\begin{aligned}\Delta_{\mathfrak{A}_n}^{\mathfrak{E}_e}(z) &= (z + d_{EE}\lambda_n + \delta_E) \cdot \left\{ \left(z + d_{II}\lambda_n - \frac{(\beta + \delta_I + \kappa)(-a + \beta + \delta_I + \kappa)}{a} \right) \right. \\ &\quad \cdot \left(z + d_{SS}\lambda_n + \frac{a^2 + a(-2(\beta + \delta_I + \kappa) + \delta_S + \psi) + (\beta + \delta_I + \kappa)^2}{a} \right) \\ &\quad \left. - \frac{(-a + \beta + \delta_I + \kappa)^2 \left(a\beta - (\beta + \delta_I + \kappa)^2\right)}{a^2} \right\} =: (z + d_{EE}\lambda_n + \delta_E) \cdot P(z)\end{aligned}$$

and the stability of (13) depends only on the stability of the quadratic polynomial P for which

$$\Delta_{\mathfrak{A}_n}(z) \equiv (z + d_{EE}\lambda_n + \delta_E) \cdot P(z),$$

resp.

$$\begin{aligned}
P(z) &= \left(z + d_{II}\lambda_n - \frac{(\beta+\delta_I+\kappa)(-a+\beta+\delta_I+\kappa)}{a}\right) \\
&\quad \cdot \left(z + d_{SS}\lambda_n + \frac{a^2 + a(-2(\beta+\delta_I+\kappa)+\delta_S+\psi)+(\beta+\delta_I+\kappa)^2}{a}\right) \\
&\quad - \frac{(-a+\beta+\delta_I+\kappa)^2\left(a\beta-(\beta+\delta_I+\kappa)^2\right)}{a^2} \\
&= z^2 + z(a-\beta-\delta_I-\kappa+\delta_S+d_{II}\lambda_n+d_{SS}\lambda_n+\psi) \\
&\quad + \frac{1}{a^2}\Bigg(\left((-a+\beta+\delta_I+\kappa)^2\left(a\beta-(\beta+\delta_I+\kappa)^2\right)\right. \\
&\quad + \left(-(\beta+\delta_I+\kappa)^2 + a(\beta+\delta_I+\kappa+d_{II}\lambda_n)\right) \\
&\quad \cdot \left(a^2 + a(-2(\beta+\delta_I+\kappa)+\delta_S+\psi+d_{SS}\lambda_n)+(\beta+\delta_I+\kappa)^2\right)\Bigg)
\end{aligned}$$

hold. Due to the condition (19) the coefficient of the linear term is clearly positive:

$$a-\beta-\delta_I-\kappa+\delta_S+d_{II}\lambda_n+\psi>0,$$

therefore we have only to show that the constant term

$$\begin{aligned}
C &:= \frac{1}{a^2}\Bigg(\left((-a+\beta+\delta_I+\kappa)^2\left(a\beta-(\beta+\delta_I+\kappa)^2\right)+\left(-(\beta+\delta_I+\kappa)^2+a(\beta+\delta_I+\kappa+d_{II}\lambda_n\right)\right) \\
&\quad \cdot \left(a^2+a(-2(\beta+\delta_I+\kappa)+\delta_S+\psi+d_{SS}\lambda_n)+(\beta+\delta_I+\kappa)^2\right)\Bigg).
\end{aligned}$$

is also positive. Using the notations

$$A := -a+\beta+\delta_I+\kappa<0 \qquad \text{and} \qquad B := \beta+\delta_I+\kappa>0$$

we get

$$\begin{aligned}
C &= \frac{1}{a^2}\left(-A^2(a\beta-B^2)+\left(-B^2+a(d_{II}\lambda_n+B)\right)\cdot\left(A^2+a\delta_S+ad_{II}\lambda_n+a\psi\right)\right) \\
&= \frac{1}{a^2}\left(-A^2(a\beta-B^2)+A^2\left(a(d_{II}\lambda_n+B)-B^2\right)\right. \\
&\quad \left.+\left(-B^2+a(d_{II}\lambda_n+B)\right)\cdot(a\delta_S+ad_{II}\lambda_n+a\psi)\right)\geq 0
\end{aligned}$$

Because

$$a\delta_S+ad_{II}\lambda_n+a\psi\geq 0 \qquad \text{and} \qquad -B^2+a(d_{II}\lambda_n+B)\geq 0 \qquad \text{and} \qquad a\beta-B^2\leq a(d_{II}\lambda_n+B)-B^2,$$

we have proven the following

Theorem. If the endemic equilibrium $\mathfrak{E}_e$ exists, i.e. condition (19) holds then the $\mathfrak{E}_e$ as equilibrium of system (3) remains stable.

This means that diffusion can't cause in this case instability.

In order to have diffusion driven instability we assume that cross-diffusion is present, as well, i.e. the matrix D in (3) has the form

$$D := \begin{bmatrix} d_{SS} & d_{SE} & 0 \\ d_{ES} & d_{EE} & 0 \\ d_{IS} & d_{IE} & d_{II} \end{bmatrix}. \tag{21}$$

In order to have Hopf bifurcation one has to show that a pair of complex conjugate roots of the corresponding characteristic polnomial

$$\mu(h) \pm \imath\nu(h)$$

crosses the imaginary axis with non-zero velocity, that is for a $h_* > 0$

$$\mu(h_*) = 0, \qquad \nu(h_*) \neq 0 \qquad \text{and} \qquad \mu'(h_*) \neq 0$$

hold. This is fulfilled (cf. [8]) if exists $n \in \mathbb{N}_0$ and $h_* > 0$ such that

$$\mathfrak{T}_n(h^*) \neq 0, \qquad \widetilde{\mathfrak{A}}_n(h^*) < 0, \qquad \mathfrak{D}_n(h^*) = \mathfrak{T}_n(h^*) \cdot \widetilde{\mathfrak{A}}_n(h^*) \tag{22}$$

and

$$\frac{d}{dh}\left\{\mathfrak{T}_n(h) \cdot \widetilde{\mathfrak{A}}_n(h) - \mathfrak{D}_n(h)\right\}\Big|_{h=h^*} \neq 0. \tag{23}$$

For the bifurcation parameter we choose $h := d_{SE}$. Using the notations

$$A := -a+\beta+\delta_I+\kappa < 0, \qquad B := \beta+\delta_I+\kappa > 0, \qquad K := (a-\beta+\delta_E-\delta_I+\delta_S-\kappa+(d_{SS}+d_{EE}+d_{II})\lambda+\psi) > 0$$

and by solving the third equation in (22) for d_{SE}, we get $d_{SE}^* = Z/W$ where

$$\begin{aligned}
Z \;:=\;& (a\beta - B^2)(\delta_E + d_{EE}\lambda_n)(-A^2 - a d_{IS}\lambda_n) + a^2 d_{IE}\kappa\lambda_n \cdot K - (a\beta - B^2)(-A^2 - a d_{IS}\lambda_n) \cdot K \\
& -(\delta_E + d_{EE}\lambda_n)(-B^2 + a(B + d_{II}\lambda_n)(A^2 + a(d_{SS}\lambda_n + \delta_S + \psi)) \\
& +a(\delta_E + d_{EE}\lambda_n) \cdot K(A^2 + a(d_{SS}\lambda_n + \delta_S + \psi)) + (-B^2 + a(B + d_{II}\lambda_n)) \\
& \cdot K \cdot (A^2 + a(d_{SS}\lambda_n + \delta_S + \psi)) + a d_{IE}\lambda_n \Big((a\beta - B^2)(d_{ES}\lambda_n - \psi) - \kappa(A^2 + a(d_{SS}\lambda_n + \delta_S + \psi))\Big), \\
W \;:=\;& a^2\lambda_n \left(\kappa\left(\frac{(-a+\beta+\delta_I+\kappa)^2}{a} - d_{IS}\lambda_n\right)\right. \\
& \left. +(\psi - d_{ES}\lambda_n)\left(\frac{(\beta+\delta_I+\kappa)(-a+\beta+\delta_I+\kappa)}{a} - d_{II}\lambda_n\right) + (-\psi + d_{ES}\lambda_n) \cdot K\right).
\end{aligned}$$

Clearly $\mathfrak{T}_n(d_{SE}^*) \neq 0$. By examining W, Z, we can see that for d_{SE}^* to be positive, the following conditions are sufficient:

$$W \geq 0, \qquad a\beta - B^2, \qquad a d_{IE}\lambda_n\kappa < a(\delta_E + d_{EE}\lambda_n) \cdot K, \qquad d_{ES}\lambda_n - \psi < 0. \tag{24}$$

If d_{SE}^* is positive, then the positivity of $B^{\mathfrak{E}_e}$ is guaranteed if additionally

$$\frac{BA}{a} - d_{II}\lambda_n < 0 \qquad \text{and} \qquad \frac{A^2}{a} - d_{IS}\lambda_n > 0 \tag{25}$$

are fulfilled. Lastly, we need the derivative to not disappear:

$$\frac{A^2}{a} - d_{IS}\lambda_n \neq \left(\frac{BA}{a} - d_{II}\right)(-d_{ES}\lambda_n + \psi) + (d_{ES}\lambda_n - \psi) \cdot K. \tag{26}$$

If conditions (24) - (26) hold, then the transversality conditions are fulfilled, therefore Turing-Hopf bifurcation takes place when the parameter d_{SE} crosses the critical value d_{SE}^*. An example of parameters that fulfil conditions (24)–(26):

$$\begin{aligned}
&a := 3, \quad \beta := 1, \quad \delta_I := \frac{1}{2}, \quad \delta_E := \frac{1}{4}, \quad \delta_S := \frac{1}{64}, \quad \kappa := 1, \quad \psi := \frac{1}{8}, \\
&d_{SS} := \frac{1}{64}, \quad d_{EE} := \frac{1}{32}, \quad d_{II} := 1, \quad d_{ES} := \frac{1}{64}, \quad d_{IE} := \frac{1}{4}, \quad d_{IS} := \frac{1}{8192}.
\end{aligned}$$

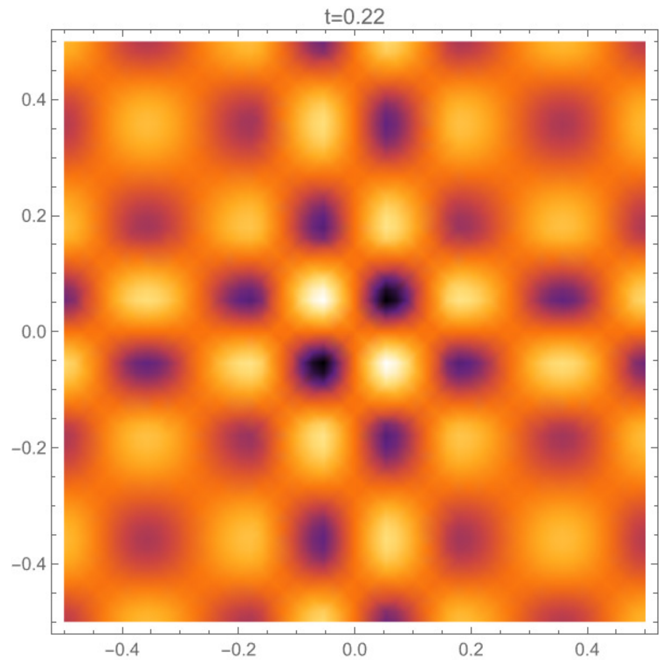

Figure 1: The first component of the solution of system (3) when (24) - (26) hold.

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