Stability and bifurcations in a reaction-diffusion system modelling disease propagation

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Summary

The kinetic system

SIS-modell

In [10] the following SIS epidemic model was proposed:

$$\left.\begin{aligned} \dot{S} &:= \lambda - \frac{aSI}{S+I} + \beta I - \psi S - \delta_S S, \\ \dot{E} &:= \psi S + \kappa I - \delta_E E, \\ \dot{I} &:= \frac{aSI}{S+I} - \kappa I - \beta I - \delta_I I. \end{aligned}\right\} \quad (1)$$

- $\delta_k > 0$: death rates,
- $\lambda > 0$: birth rate,
- $a > 0$: transmission coefficient,
- $\beta > 0$: recovery rate,
- $\kappa > 0$ and $\psi > 0$: educational rate of the infecteds and susceptibles.

Steady states of the kinetic system

In [10] the authors

1. calculated the basic reproduction ratio

$$\mathcal{R}_0 := \frac{a}{\kappa + \beta + \delta_I}.$$

- In case of $\mathcal{R}_0 < 1$ the system has one asymptotically stable disease free steady state;
- In case of $\mathcal{R}_0 > 1$, there are two strady states $\mathfrak{E}_b$ and $\mathfrak{E}_e$

2. proved that in case of $\mathcal{R}_0 = 1$ the steady state $\mathfrak{E}_b$ loses its stability through a transcritical bifurcation, exchanging stability with the new equlibrium $\mathfrak{E}_e$.

Adding diffusion to the system

The reaction-diffusion system

$$\left.\begin{array}{ll} \partial_t \mathrm{u} = D \cdot \Delta_{\mathrm{r}} \mathrm{u} + \mathrm{f}(\mathrm{u}) & \text{in } \Omega \times \mathbb{R}_0^+, \\ (\mathrm{n} \cdot \nabla_{\mathrm{r}}) \mathrm{u}(\mathrm{r}, t) = 0 & ((\mathrm{r}, t) \in \partial\Omega \times \mathbb{R}_0^+), \\ \mathrm{u}(\mathrm{r}, 0) = \mathrm{u}_0(\mathrm{r}) & ((\mathrm{r}, t) \in \overline{\Omega} \times \{0\}) \end{array}\right\} \qquad (2)$$

where

$$D := \begin{bmatrix} d_{SS} & d_{SE} & 0 \\ d_{ES} & d_{EE} & 0 \\ d_{IS} & d_{IE} & d_{II} \end{bmatrix}.$$

- $\mathfrak{E}_b$ and $\mathfrak{E}_e$ are steady states of system (2), too.
- In [10] the authors showed, that the positive octant of the phaseplane is an invariant set for (1)

Positivity with self-diffusion

If

$$\mathbf{\Phi} = (\Phi_1, \Phi_2, \Phi_3) : \overline{\Omega} \times \mathbb{R}_0^+ \to \mathbb{R}^3$$

is a solution of (2), then

- using a theorem from [5] and
- observing that $\Phi_3 \equiv 0$ is a solution of the third equation in (2) we get:

Theorem

If D is a positive diagonal matrix then all solutions $\mathbf{\Phi} = (\Phi_1, \Phi_2, \Phi_3) \in \overline{\Omega} \times \mathbb{R}_0^+ \to \mathbb{R}^3$ of (2) with positive initial values $\Phi_1(0) > 0$, $\Phi_2(0) > 0$, $\Phi_3(0) > 0$ remain positive for all $t \geq 0$ in their domain of existence.

Dissipativity

Theorem

If D is a positive scalar matrix then system (2) is dissipative.

Theorem

If D is a positive diagonal matrix then condition

$$\psi + \delta_S = \kappa + \delta_I =: \mu \tag{3}$$

implies that system (2) is dissipative.

Linearization 1.

Let $\mathfrak{E}^* \in \{\mathfrak{E}_b, \mathfrak{E}_e\}$, then the linarization of system (2) at $\mathfrak{E}^*$ has the form:

$$\left.\begin{aligned} \partial_t \mathsf{v} &= D \cdot \Delta_{\mathsf{r}} \mathsf{v} + \mathfrak{A}\mathsf{v} && \text{in } \Omega \times \mathbb{R}_0^+, \\ (\mathsf{n} \cdot \nabla_{\mathsf{r}})\, \mathsf{v}(\mathsf{r}, t) &= 0 && ((\mathsf{r}, t) \in \partial\Omega \times \mathbb{R}_0^+), \\ \mathsf{v}(\mathsf{r}, 0) &= \mathsf{v}_0(\mathsf{r}) && ((\mathsf{r}, t) \in \overline{\Omega} \times \{0\}) \end{aligned}\right\} \tag{4}$$

where

$$\mathfrak{A} := \mathsf{f}'(\mathfrak{E}^*).$$

Linearization 2.

Solving system (4) using Fourier-method (c.f. [11]):

$$\mathrm{v}(\mathrm{r}, t) = \sum_{\nu=0}^{\infty} \psi_\nu(\mathrm{r}) \exp\left(\mathfrak{A}_\nu t\right) \mathbf{\Lambda}_\nu,$$

where for any $\nu \in \mathbb{N}_0$:

$$\mathfrak{A}_\nu := \mathrm{f}'(\mathfrak{E}^*) - \lambda_\nu D, \qquad \mathbf{\Lambda}_\nu := \int_\Omega \mathrm{v}_0(\mathrm{r}) \psi_\nu(\mathrm{r})\, \mathrm{dr},$$

resp. λ_ν and ψ_ν are eigenvalues and eigenfunctions of $-\Delta_\mathrm{r}$ with HNBC:

$$\Delta_\mathrm{r} \psi_\nu = -\lambda_\nu \psi_\nu, \qquad \partial_\mathrm{n} \psi_\nu |_{\partial\Omega} = 0.$$

It can be proven that

$$0 = \lambda_0 \le \lambda_1 \le \lambda_2 \le \ldots \le \lambda_\nu \longrightarrow +\infty \qquad (n \to \infty).$$

Diffusional instability - only self-diffusion

Examining the characteristic polynomial of $\mathfrak{A}_\nu$ we get

Theorem

- In case $\mathcal{R}_0 < 1$, the steady state $\mathfrak{E}_b$ cannot lose its stability.
- In case $\mathcal{R}_0 > 1$, diffusion stabilizes $\mathfrak{E}_b$, if

$$d_{II} > \frac{a - (\beta + \delta_I + \kappa)}{\lambda_1} \tag{5}$$

holds.

Theorem

If the steady state $\mathfrak{E}_e$ exists, then it remains asymptotically stable for all diagonal matrices D.

Diffusional instability of $\mathfrak{E}_b$ - with cross-diffusion 1.

Examining the cahracterictic polynomial in this case too, we get

Theorem

In case $\mathcal{R}_0 < 1$, if

$$d_{ES} d_{SE} > d_{EE} d_{SS}$$

holds, then the steady state $\mathfrak{E}_b$ loses its stability.

Diffusional instability of $\mathfrak{E}_b$ - with cross-diffusion 2.

Let introduce

$$
\begin{aligned}
w &:= d_{ES}d_{SE} - d_{EE}d_{SS},\\
q &:= d_{ES}\psi - d_{EE}\delta_S - d_{EE}\psi - \delta_E d_{SS},\\
z &:= -\delta_E\delta_S - \delta_E\psi.
\end{aligned}
$$

Theorem

In case $\mathcal{R}_0 > 1$, if

$$d_{II} > \frac{a - (\beta + \delta_I + \kappa)}{\lambda_1}, \quad w < 0 \qquad \text{and} \qquad q^2 - 4wz < 0$$

hold, then diffusion stabilizes the steady state $\mathfrak{E}_b$.

Diffusional instability of $\mathfrak{E}_e$ - with cross-diffusion 1.

The characteristic polynomial $\mathfrak{A}_\nu$:

$$\Delta_{\mathfrak{A}_\nu}(z) := z^3 - \mathfrak{T}_\nu z^2 + \widetilde{\mathfrak{A}}_\nu z - \mathfrak{D}_\nu \qquad (z \in \mathbb{C})$$

In order to have Hopf bifurcation one has to show that a pair of complex conjugate roots

$$\mu(h) \pm \imath\nu(h)$$

crosses the imaginary axis with non-zero velocity.

Diffusional instability of $\mathfrak{E}_e$ - with cross-diffusion 2.

This is fulfilled if for a $h_* > 0$ there exists $\nu \in \mathbb{N}_0$ such that

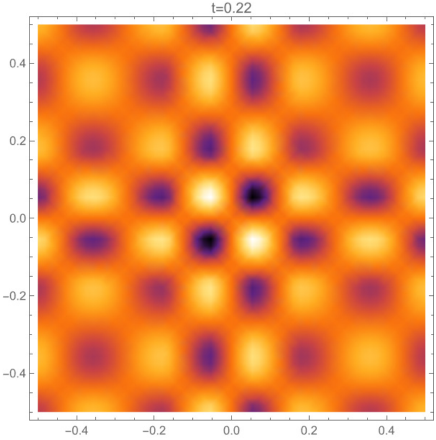
$$\mathfrak{T}_\nu(h^*) \neq 0, \qquad \widetilde{\mathfrak{A}}_\nu(h^*) < 0, \qquad \mathfrak{D}_\nu(h^*) = \mathfrak{T}_\nu(h^*) \cdot \widetilde{\mathfrak{A}}_\nu(h^*) \quad (6)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}h} \left\{ \mathfrak{T}_\nu(h) \cdot \widetilde{\mathfrak{A}}_\nu(h) - \mathfrak{D}_\nu(h) \right\} \Big|_{h=h^*} \neq 0. \quad (7)$$

We chose d_{SE} as bifurcation parameter and

- showed the existence of Turing-Hopf bifurcation under some conditions regarding the system parameters,
- gave an example of parameters and made a simulation with MATHEMATICA®.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Thank you for your attention!

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