Tensegrity frameworks

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This semester, I started exploring a completely new topic, the rigidity of tensegrity frameworks, with my supervisor, Tibor Jordán. The beginning of the semester was dedicated to understanding the concepts and basic previous results of the field, summarized in Chapter 1. Fortunately, in a relatively short time I was able to understand and investigate some open problems. I present the developments related to these in this report. Theorem 2.2 in Chapter 2 is a new result from the work of this semester, where we managed to prove a sharp upper bound on the number of edges for minimally infinitesimally rigid tensegrity frameworks in d-dimensions (the case $d = 1$ was previously settled in [4]). In Chapter 3, I summarize our our steps and plan towards proving a related conjecture (Conjecture 3.1).

One semester is not a long time, so most of the questions remained open. We hope to answer some of them in the next few months. Since I will also write my master thesis on this topic, this is not a classic final report; the work continues after submission, hopefully with more results.

1 Introduction

In rigidity theory, we usually deal with frameworks where the vertices of a graph are considered as joints and the edges as rigid bars. Therefore, we consider a motion of the framework legal, if the distance between the two endpoints of each edge remains unchanged.

Tensegrity frameworks generalize bar frameworks with cables and struts instead of bars. Cables cannot be stretched, and struts cannot be compressed; however, motion in the other direction is permitted between their endpoints. The interest in tensegrity frameworks was significantly raised by the works of the sculptor Kenneth Snelson. His monumental creations, consisting of bars and cables, are visually striking as they stably stand with the bars not touching each other, creating an illusion of them floating in the air [8]. Our focus on tensegrity frameworks is less artistic and more concerned with rigidity theory. First we introduce the fundamental definitions about tensegrities. We use the sources [1, 4, 5, 6, 7].

A tensegrity graph $T = (V, C \cup S)$ is a graph on vertex set V, in which each edge e is labelled as a cable or a strut. Accordingly, the edge set of T is partitioned into two sets, C and S.

A d-dimensional tensegrity framework (T, p) is a pair, where $T = (V, C \cup S)$ is a tensegrity graph and $p : V \longrightarrow \mathbb{R}^d$ is a map, satisfying $p(u) \neq p(v)$ for each $uv \in C \cup S$. We also say that (T, p) is a realization of T in $\mathbb{R}^d$.

A d-dimensional realization (T, p) of T is called generic if the set of the $d|V|$ coordinates of the points $p(v)$, $v \in V(T)$, is algebraically independent over the rationals. It is injective if $p(u) \neq p(u)$ for all pairs of distinct vertices $u, v \in V$.

The underlying graph of $T = (V, C \cup S)$, denoted by $\overline{T} = (V, E)$, is a graph on vertex set V in which $uv \in E$ if and only if $uv \in C \cup S$ holds. The bar-and-joint framework $(\overline{T}, p)$ is the framework obtained by replacing every strut and cable in (T, p) with a bar.

An infinitesimal motion of a tensegrity framework (T, p) is an assignment $m : V \longrightarrow \mathbb{R}^d$ which satisfy

$$(p(u) - p(v)) \cdot (m(u) - m(v)) \leq 0 \quad \text{for each cable } uv \in C,$$
$$(p(u) - p(v)) \cdot (m(u) - m(v)) \geq 0 \quad \text{for each strut } uv \in S.$$

A tensegrity framework (T, p) is infinitesimally rigid if every infinitesimal motion of (T, p) is an infinitesimal isometry of $\mathbb{R}^d$.

The matrix of this system of linear inequalities is the rigidity matrix $R(T, p)$ of (T, p) of size $|E| \times d|V|$, where, for each edge $e = uv \in C \cup S$, in the row corresponding to e, the entries in the two columns corresponding to vertices u and v contain the d coordinates of $(p(u) - p(v))$ and $(p(v) - p(u))$, respectively, and the remaining entries are zeros. Notice that the rigidity matrix of a tensegrity framework (T, p) is the same as the rigidity matrix of the bar-and-joint framework $(\overline{T}, p)$. It is known that a bar-and-joint framework (G, p) is infinitesimally rigid if and only if the rank of its rigidity matrix is $S(|V|, d)$, where

$$S(|V|, d) = \begin{cases} d|V| - \binom{d+1}{2} & \text{if } |V| \geq d + 2 \\ \binom{|V|}{2} & \text{if } |V| \leq d + 1 \end{cases}$$

For a set A of edges of T we use $R_A(T, p)$ to denote the submatrix of the rigidity matrix $R(T, p)$ induced by the rows of A. Thus, $m \in \mathbb{R}^{d|V|}$ is a infinitesimal motion of a tensegrity framework (T, p) if $R_C(T, p) \cdot m \leq 0$ and $R_S(T, p) \cdot m \geq 0$.

For the infinitesimal rigidity of tensegrity frameworks Roth and Whiteley gave a nice characterizaton (Theorem 1.1), in which they use the concept of a stress on the edge set of a framework.

A stress of a tensegrity framework (T, p) in $\mathbb{R}^d$ is a function $\omega : C \cup S \to \mathbb{R}$, which assigns a scalar to each edge of T such that $\omega(e) \le 0$ for each cable $e \in C$, $\omega(e) \ge 0$ for each strut $e \in S$ and

$$\sum_{uv \in C \cup S} \omega(uv)(p(u) - p(v)) = 0 \text{ for each vertex } v \in V.$$

Notice that it means $\omega \cdot R(T, p) = 0$.

The support of a stress ω of (T, p) is the set of edges with non-zero stress, i.e. $\text{supp}(\omega) = \{e \in C \cup S : \omega(e) \neq 0\}$. A proper stress ω of a tensegrity framework (T, p) in $\mathbb{R}^d$ is a stress of (T, p) with every edge on its support, i.e. $\omega(e) < 0$ for each cable $e \in C$, $\omega(e) > 0$ for each strut $e \in S$

An infinitesimally rigid tensegrity framework (T, p) in $\mathbb{R}^d$ is called minimally infinitesimally rigid in $\mathbb{R}^d$ if $(T - e, p)$ is not infinitesimally rigid in $\mathbb{R}^d$ for every edge e of T.

Now we introduce a fundamental theorem of the study of tensegrity frameworks, attributed to Roth and Whiteley [1].

Theorem 1.1 (Roth, Whiteley)**.** Let (T, p) be a tensegrity framework in $\mathbb{R}^d$. Then (T, p) is infinitesimally rigid if and only if $(\overline{T}, p)$ is infinitesimally rigid and there exists a proper stress of (T, p).

In the following chapters, we discuss minimally infinitesimally rigid tensegrity frameworks. Note that if a d-dimensional tensegrity framework (T, p) has no more than $d + 1$ vertices, then, according to Theorem 1.1 and our knowledge on bar frameworks, if (T, p) is minimally infinitesimally rigid, then T can only be the cable-strut complete graph (i.e., $(V, C) = (V, S) = K_{|V|}$). In this case, we understand the minimal instances well, and they are not very interesting. Therefore, in the following, we will only consider cases where the number of vertices is at least $d + 2$.

2 Minimally infinitesimally rigid tensegrity frameworks

Consider a minimally infinitesimally rigid framework (G, p) in $\mathbb{R}^d$. Removing any edge uv from (G, p), there exists an infinitesimal motion where the distance between u and v increases and another one where it decreases (because the negative of an infinitesimal motion is also a motion). Therefore, replacing each bar in (G, p) with a parallel strut-cable pair leads to an infinitesimally rigid tensegrity (it is intuitively clear and also easy to verify using the conditions in Theorem 1.1) and removing any edge from this tensegrity allows an infinitesimal motion. Thus, tensegrity frameworks constructed this way are minimally infinitesimally rigid. It is known that a d-dimensional minimally infinitesimally rigid framework with $|V| \geq d+2$ has exactly $d|V| - \binom{d+1}{2}$ edges [5], implying the existence of a minimally infinitesimally rigid tensegrity framework in $\mathbb{R}^d$ with $2 \cdot (d|V| - \binom{d+1}{2})$ edges for any d and $|V| \geq d+2$.

We show that a minimally infinitesimally rigid tensegrity framework in $\mathbb{R}^d$ cannot have more edges than this. The proof employs a variant of Carathéodory's theorem from convex geometry, attributed to Ernst Steinitz [2], see also [3].

Let $A \subset \mathbb{R}^n$ be a finite set of points. The convex hull of A is the set $\text{conv}(A) = \{\sum \lambda_i a_i : a_i \in A, \sum \lambda_i = 1, \lambda_i \geq 0\}$.

Theorem 2.1 (Steinitz)**.** Consider $X \subset \mathbb{R}^n$ a finite set of points, where $\text{conv}(X)$ is k-dimensional and contains a k-dimensional ball centered at x. Then there is a subset $Y \subseteq X$ of at most $2k$ points such that $\text{conv}(Y)$ contains a k-dimensional ball with the same center.

Using this theorem, we prove a sharp upper bound on the edge count of minimally infinitesimally rigid tensegrity frameworks, depending on the number of vertices and dimensions.

Theorem 2.2. Let (T, p) be a minimally infinitesimally rigid realization of $T = (V, C \cup S)$ in $\mathbb{R}^d$ with $|V| \geq d+2$. Then $|C \cup S| \leq 2 \cdot \left(d|V| - \binom{d+1}{2}\right)$.

Proof. Assume that (T, p) is minimally infinitesimally rigid in $\mathbb{R}^d$. Let $N = d|V| - \binom{d+1}{2}$. Let $R'(T, p)$ be the matrix obtained by replacing the rows corresponding to cables in the rigidity matrix $R(T, p)$ with their negatives. According to Theorem 1.1 by Roth and Whiteley, the tensegrity framework (T, p) is infinitesimally rigid if and only if

(1) $\overline{T}$ is infinitesimally rigid, equivalently: the rows of $R'(T, p)$ generate a subspace of dimension N,

(2) and there exists $\omega \in \mathbb{R}^E$ such that $\omega > 0$ and $\omega \cdot R'(T, p) = 0$, equivalently: there exists a positive convex combination of the rows of $R'(T, p)$ that results in zero.

Let $X \subset \mathbb{R}^{d|V|}$ denote the set of the rows of $R'(G, p)$. Then (1) and (2) implies that (T, p) is infinitesimally rigid if and only if $\text{conv}(X)$ forms an N-dimensional polytope containing an N-dimensional ball centered at the origin. By Steinitz's theorem, one can select a set $Y \subseteq X$ with at most $2N$ elements such that $\text{conv}(Y)$ forms a N-dimensional polytope containing a N-dimensional ball centered at the origin.

Therefore, if $|C \cup S| \geq 2N + 1$ then there exists a row of $R'(T, p)$ such that removing it does not change the rank of the matrix and there exists a positive convex combination of the rows that results in zero. So, by removing an edge, (1) and (2) still hold, thus the framework remains infinitesimally rigid, contradicting the minimality of T. □

3 Without parallel edges

A further interesting question is whether we can prove a better upper bound on the number of edges in minimally infinitesimally rigid tensegrity frameworks when we forbid parallel cable-strut pairs.

In the 1-dimensional case, we already know the answer. According to [4], the bound of Theorem 2.2 is sharp (i.e. a minimally rigid tensegrity framework has exactly $2|V| - 2$ edges) in $\mathbb{R}^1$ if and only if the framework is a cable-strut tree. Therefore, if (T, p) is a minimally rigid realization of $T = (V, C \cup S)$ in $\mathbb{R}^1$, and there are no parallel edges in T, then $|C \cup S| \leq 2|V| - 3$. This is sharp: consider the tensegrity T that consists of a complete bipartite graph $K_{2,|V|-2}$ of cables and an additional strut connecting the two vertices of degree $(|V| - 2)$. It is easy to see that T has a minimally infinitesimally rigid representation on the line with exactly $2|V| - 3$ edges.

In the 2-dimensional case, the question is currently open. We conjecture that it remains true here (and in the d-dimensional case in general), that the bound of Theorem 2.2 will be sharp precisely for those tensegrities that correspond to minimally infinitesimally rigid bar-and-joint frameworks, where the bars are replaced by parallel cable-strut pairs.

Consider the following tensegrity framework (T, p) in $\mathbb{R}^2$: take a unit square of cables, and add $|V| - 4$ vertices precisely at its center, each connected to every vertex of the square with struts. It can be easily verified that (T, p) is minimally infinitesimally rigid in $\mathbb{R}^2$ and has exactly $4(|V| - 4) + 4 = 4|V| - 12$ edges, which is very close to the bound proved in Theorem 2.2. However, notice that this example is not generic (in fact, it is not even injective), and if we slightly move any of the central points (assume that there are at least two of them), then one of the struts attached to it becomes redundant, making the framework no longer minimal. Therefore, without any restrictions on p, we cannot hope for a significantly better upper bound.

Now consider the case where p is assumed to be generic. In this scenario, our conjecture for the upper bound in the 2-dimensional generic case is $3|V| - 6$ (the general conjecture for d-dimension is $(d + 1)|V| - \binom{d+2}{2}$ according to [4]). During most of the semester, we attempted to verify this. In this chapter, we summarize the observations related to this question, which hopefully bring us closer to the proof.

Conjecture 3.1. Let (T, p) be a minimally infinitesimally rigid generic realization of $T = (V, C \cup S)$ in $\mathbb{R}^2$ with $|V| \geq d + 2$. Then $|C \cup S| \leq 3|V| - 6$.

If this conjecture was true, then it is sharp. Consider the following tensegrity T: take a triangle of cables, and add $|V| - 3$ vertices, each connected to every vertex of the triangle with struts. It is easy to see that T has a minimally infinitesimally rigid generic realization in $\mathbb{R}^2$ and has exactly $3|V| - 6$ edges.

As we mentioned before, Clay, Jordán, and Palmer proved Theorem 2.2 for the case of $d = 1$ in [4]. One key idea of the proof was using Lemma 3.1 from the paper of Roth and Whiteley [1]. The proof involves several steps that can be generalized to two or even higher dimensions. We hope that by adding some new ideas, we can provide a similar proof for Conjecture 3.1.

We plan to apply some matroid theory, we use the terminology of the technical report of Jordán [5].

The rigidity matroid of a d-dimensional framework (G, p) is defined on the edge set E of G, where $F \subseteq E$ is independent if and only if the corresponding rows of the rigidity matrix $R(G, p)$ are linearly independent. It is known that if p is generic, then the rigidity matroid depends only on G. Therefore, we denote the rigidity matroid associated with a d-dimensional generic realization of G by $\mathcal{R}_d(G)$. It is not difficult to see that $\mathcal{R}_1(G)$ is the circuit matroid of G. According to Laman's theorem, $\mathcal{R}_2(G)$ is also well-characterized, since it is equivalent to the sparsity matroid of G. More precisely, the edge set of a subgraph H of G is independent in the rigidity matroid if and only if H is sparse, i.e. for every subset X of at least 2 vertices in H, the number of edges spanned by X is at most $2|X| - 3$.

A subgraph $H = (W, C)$ of G is said to be an M-circuit in G if C is a circuit (i.e. a minimal dependent set) in $\mathcal{R}_2(G)$, equivalently: $|C| = 2|W| - 2$, and for all proper subsets $X \subset W$ there are at most $2|X| - 3$ edges in C induced by X.

A matroid is connected if, for any two of its elements, there exists a circuit containing both of them. Connectivity of the matroid can be characterized using matroid ear decomposition. We say that a sequence of circuits $C_1, C_2, \ldots, C_t$ of the matroid $\mathcal{M} = (E, \mathcal{I})$ is an ear decomposition of $\mathcal{M}$ if $C_1 \cup C_2 \cup \ldots \cup C_t = E$ and for all $2 \leq i \leq t$ the following properties hold: (E1) $C_i \cap (C_1 \cup C_2 \cup \ldots \cup C_{i-1}) \neq \emptyset$, (E2) $C_i - (C_1 \cup C_2 \cup \ldots \cup C_{i-1}) \neq \emptyset$ and C_i is minimal with respect to (E1) and (E2), i.e. no circuit C_i' satisfying (E1) and (E2) has $C_i' - (C_1 \cup C_2 \cup \ldots \cup C_{i-1})$ properly contained in $C_i - (C_1 \cup C_2 \cup \ldots \cup C_{i-1})$. It is easy to verify that a matroid is connected if and only if it has a ear decomposition. We say that a graph $G = (V, E)$ is M-connected if $\mathcal{R}_2(G)$ is connected.

The following lemma from the paper of Roth and Whiteley [1] seems to be very useful.

Lemma 3.1 (Roth and Whiteley [1]). Let (T, p) be a realization of the tensegrity graph

$T = (V, C \cup S)$ in $\mathbb{R}^d$ and let $e \in C \cup S$. If there exists a stress of (T, p) with e in its support, then there exists a stress ω of (T, p) with e in its support and such that $\text{rank} R_A(T, p) = |A|$ for every $A \subset \text{supp}(\omega)$.

According to Theorem 1.1, if (T, p) is an infinitesimally rigid tensegrity framework then there exists a proper stress of (T, p), therefore, for every edge, the conditions of Lemma 3.1 are satisfied. Hence, each edge is part of a sub-tensegrity whose edge set is the support of a stress and forms a minimally connected set of rows in the rigidity matrix. In $\mathbb{R}^2$ this means that each edge of T is in an M-circuit with a proper stress on its edge set. Using Theorem 1.1 with the fact that bar-and-joint M-circuits are infinitesimally rigid, it follows that every sub-tensegrity given by Lemma 3.1 are rigid in $\mathbb{R}^2$, in fact, according to the following lemma from [4], they are minimally infinitesimally rigid, if (T, p) is minimally infinitesimally rigid.

Lemma 3.2. Let (T, p) be a minimally infinitesimally rigid tensegrity framework in $\mathbb{R}^d$. Then every infinitesimally rigid subframework of (T, p) is minimally infinitesimally rigid.

Using these lemmas, we can easily prove that the M-connected components of a minimally infinitesimally rigid tensegrity framework in $\mathbb{R}^2$ are also minimally infinitesimally rigid.

Lemma 3.3. If (T, p) is a minimally infinitesimally rigid generic tensegrity framework in $\mathbb{R}^2$, then each subframework induced by an M-connected component of T is minimally infinitesimally rigid.

Proof. Let (H, p_H) be an M-connected component of a minimally infinitesimally rigid tensegrity framework (T, p). By Lemma 3.2, it is sufficient to prove that (H, p_H) is infinitesimally rigid. Since (T, p) is infinitesimally rigid, according to Lemma 3.1, for each edge e, it is an element of an infinitesimally rigid M-circuit C_e. Note that, by the definition of matroid ear decomposition, each M-circuit of T is entirely contained in one of the M-connected components of T. Therefore, for each edge e of H the union of the corresponding M-circuits C_e results in H. Since each C_e is an infinitesimally rigid sub-tensegrity, by Theorem 1.1, there exists a proper stress on their edges, and by summing these stresses, we obtain a proper stress on (H, p_H). It is known that M-connected components are infinitesimally rigid as bar-and-joint frameworks, so the lemma follows from Theorem 1.1. □

It can be easily calculated that if the conjecture holds for the M-connected components, then it holds for the entire tensegrity framework.

Lemma 3.4. If each M-connected component of a generic tensegrity framework (T, p) satisfies the upper bound of Conjecture 3.1, then T satisfies the upper bound of Conjecture 3.1.

Proof. Let $H_1, H_2, \dots, H_q$ be the M-connected components of T. Assume that H_i have at most $3|V(H_i)| - 6$ edges for all $i = 1, 2, \dots, q$. Let $r(H_i)$ denote the rank of the matroid $\mathcal{R}_2(H_i)$. Since M-connected bar-and-joint frameworks are infinitesimally rigid, $r(H_i) = 2|V(H_i)| - 3$ for all $i = 1, 2, \dots, q$. Then

$$|C \cup S| \leq \sum_{i=1}^{q}(3|V(H_i)| - 6) = \sum_{i=1}^{q}\left(\frac{3}{2}r(H_i) - \frac{3}{2}\right) =$$

$$= \frac{3}{2}\sum_{i=1}^{q} r(H_i) - \frac{3q}{2} = 3|V| - \frac{9}{2} - \frac{3q}{2} \leq 3|V| - 6,$$

where the last equality holds, because the rank of $\mathcal{R}_2(T)$ is $2|V| - 3$ and is equal to the sum of the rank of the M-connected components of T. □

To sum up, we can see that it would be sufficient to prove the conjecture for minimally rigid tensegrity frameworks (T, p) where $\mathcal{R}_d(T)$ is connected, therefore, it has a matroid ear decomposition. The following lemma is from [5].

Lemma 3.5. Let the M-circuit C_i $(i \geq 2)$ be an element of the ear decomposition $C_1, C_2, \dots, C_t$ of the rigidity matroid $\mathcal{R}_2(G)$ of a graph G and let V_i^+ (resp. E_i^+) denote the vertices (resp. the edges) of G contained in C_i but not in $C_1 \cup C_2, \cup \dots \cup C_{i-1}$. Then $|E_i^+| = 2|V_i^+| + 1$ holds for each $2 \leq i \leq t$.

Another important lemma from [4]:

Lemma 3.6. Let (T_1, p_1) and (T_2, p_2) be infinitesimally rigid tensegrity frameworks in $\mathbb{R}^d$ such that they share a set of d vertices in general position. Then their union is infinitesimally rigid in $\mathbb{R}^d$.

If we managed to prove that there always exists an ear decomposition of $\mathcal{R}_2(T)$ where there is no trivial ear, i.e. $|V_i^+| > 0$ for each C_i then the conjecture would follow from the simple calculation below (for simplicity, V_1^+ and E_1^+ denote the vertices and edges of C_1):

$$|C \cup S| = \sum_{i=1}^{t}|E_i^+| = \sum_{i=1}^{t}(2|V_i^+| + 1) - 3 \leq 2|V| + (|V| - 3) - 3 = 3|V| - 6,$$

where we used $|E_1^+| = 2|V_1^+| - 2$ and $t \leq |V| - 3$ based on the assumption that there is no trivial ear.

If, for example, there was always an ear decomposition such that every C_i is a minimally infinitesimally rigid sub-tensegrity – this does seem possible, as, according to Lemma 3.1, each edge is part of a minimally infinitesimally rigid M-circuit – then, by Lemma 3.2 and Lemma 3.6, at each step of the ear decomposition, the tensegrity framework would be minimally infinitesimally rigid, thus induced by a vertex set, so none of the ears could be trivial.

The results look promising; however, we currently do not see how to find such an ear decomposition. Therefore, Conjecture 3.1 remains a conjecture for now.

We can prove the conjecture if we assume that all the vertices are in convex position and the subgraph (V, C) forms the boundary cycle of the convex hull. Notice, that in this case all of the minimally infinitesimally rigid M-circuits given by Lemma 3.1 contain the subgraph (V, C) (this is because there must be at least one cable in each of the infinitesimally rigid subframeworks and it follows from the setting of the cables that if one of them is the element of an infinitesimally rigid sub-tensegrity, then all of them are). Since these circuits are minimally rigid, they are induced by their vertex sets V, so there can be only one of them. Therefore, a tensegrity framework from this family is minimally infinitesimally rigid if and only if its graph is an M-circuit.

Our next goal is to verify the conjecture for the same family with the modification that allows for points to be inside the convex hull as well, but (V, C) still forms the boundary cycle of the convex hull. Here it is also true that the minimally infinitesimally rigid circuits given by Lemma 3.1 contain (V, C), so, property (E1) of ear decomposition holds for any subset and order of the circuits which satisfly property (E2). However, it is not clear how we ensure the minimality of the ears.

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