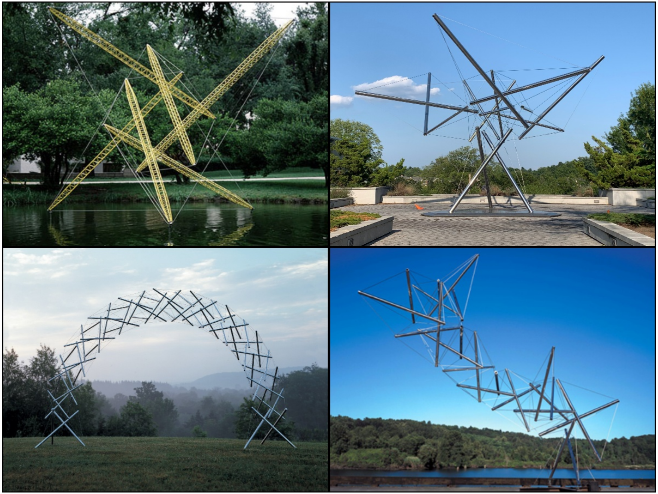
Minimally infinitesimally rigid tensegrity frameworks

Sára Hanna Tóth

Supervisor:
Tibor Jordán

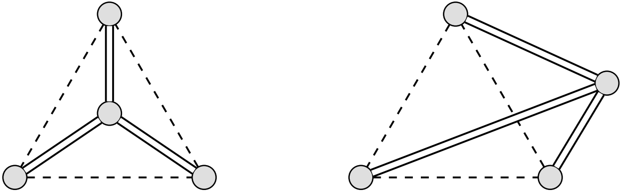
Kenneth Snelson



Tensegrity frameworks

A **tensegrity graph** $T = (V, C \cup S)$ is a graph, in which each edge is labelled as a **cable** or a **strut**.

A d-dimensional **tensegrity framework** (T, p) is a pair, where $T = (V, C \cup S)$ is a tensegrity graph and $p : V \to \mathbb{R}^d$ is a map.



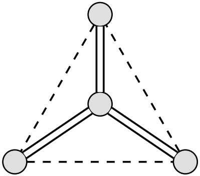
Infinitesimal rigidity

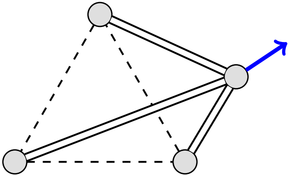
An **infinitesimal motion** of a tensegrity framework is an assignment $m : V \longrightarrow \mathbb{R}^d$ satisfying

$$(p(u) - p(v)) \cdot (m(u) - m(v)) \le 0 \text{ for each cable } uv \in C,$$
$$(p(u) - p(v)) \cdot (m(u) - m(v)) \ge 0 \text{ for each strut } uv \in S.$$

A tensegrity framework (T, p) is **infinitesimally rigid** if every infinitesimal motion of (T, p) is an infinitesimal isometry of $\mathbb{R}^d$.

An infinitesimally rigid tensegrity framework (T, p) in $\mathbb{R}^d$ is called **minimally infinitesimally rigid** in $\mathbb{R}^d$ if $(T - e, p)$ is not infinitesimally rigid in $\mathbb{R}^d$ for every edge e of T.



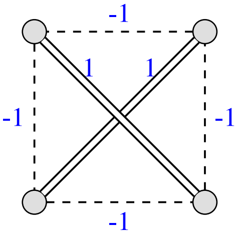


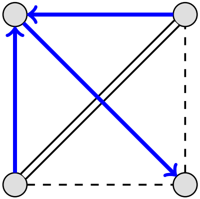
Stress

A **stress** of a tensegrity framework is a function $\omega : C \cup S \to \mathbb{R}$ such that

$$\begin{aligned} \omega(e) &\leq 0 && \text{for each cable } e \in C, \\ \omega(e) &\geq 0 && \text{for each strut } e \in S, \\ \sum_{uv \in C \cup S} \omega(uv)(p(u) - p(v)) &= 0 && \text{for each vertex } v \in V. \end{aligned}$$

A stress ω is a **proper stress** if $\omega(e) \neq 0$ for each edge $e \in C \cup S$.

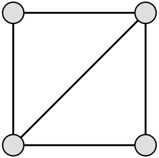




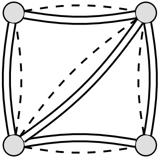
Minimally infinitesimally rigid frameworks

A tensegrity framework (T, p) is called **minimally infinitesimally rigid** if $(T - e, p)$ is not infinitesimally rigid for every edge e of T.

It is known that a d-dimensional minimally infinitesimally rigid bar-and-joint framework with $|V| \geq d + 2$ has exactly $d|V| - \binom{d+1}{2}$ edges.



(a) minimally infinitesimally rigid framework with $2|V| - 3$ bars



(b) minimally infinitesimally rigid tensegrity with $4|V| - 6$ members

Question: Can a minimally infinitesimally rigid tensegrity have more edges than $2\left(d|V| - \binom{d+1}{2}\right)$?

Our result

Question: Can a minimally infinitesimally rigid tensegrity have more edges than $2\left(d|V| - \binom{d+1}{2}\right)$?

Theorem

Let (T, p) be a minimally infinitesimally rigid realization of $T = (V, C \cup S)$ in $\mathbb{R}^d$ with $|V| \geq d + 2$. Then $|C \cup S| \leq 2 \cdot \left(d|V| - \binom{d+1}{2}\right)$.

equality holds $\iff$ (T, p) is a minimally infinitesimally rigid bar-and-joint framework with parallel cable-strut pairs instead of bars

Main idea of the proof

Theorem (Roth, Whiteley)

Let (T, p) be a tensegrity framework in $\mathbb{R}^d$. Then (T, p) is infinitesimally rigid if and only if there exists a proper stress of (T, p) and $(\overline{T}, p)$ is infinitesimally rigid.

tensegrity framework $(T, p) \leftrightarrow$ set of points $X \subset \mathbb{R}^{d|V|}$ with $|X| = |C \cup S|$

(T, p) is infinitesimally rigid $\iff$ conv(X) forms a $\left(d|V| - \binom{d+1}{2}\right)$-dimensional polytope with the origin in its relative interior

Theorem (Steinitz)

Consider $X \subset \mathbb{R}^n$ a finite set of points and a point x in the interior of conv(X). Then there is a subset $Y \subseteq X$ of at most $2n$ points such that x is in the interior of conv(Y).

Without parallel edges, $d = 1$

$|C \cup S| = 2 \cdot \left(d|V| - \binom{d+1}{2}\right) \iff (T, p)$ is a minimally infinitesimally rigid bar-and-joint framework with parallel cable-strut pairs instead of bars

Question: Can we prove a better upper bound on the number of edges if we forbid parallel cable-strut pairs?

If (T, p) is minimally infinitesimally rigid in $\mathbb{R}^1$, then $|C \cup S| \leq 2|V| - 2$.
If T has no parallel edges: $|C \cup S| \leq 2|V| - 3$. This is sharp.

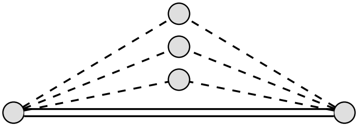
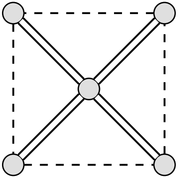


Figure: Tensegrity graph with $2|V| - 3$ edges that has a minimally infinitesimally rigid realization in $\mathbb{R}^1$.

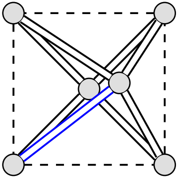
Without parallel edges, $d = 2$

If (T, p) is minimally infinitesimally rigid in $\mathbb{R}^2$, then $|C \cup S| \leq 4|V| - 6$.

Consider the following tensegrity: a unit square of cables with multiple vertices at its center, each connected to every vertex of the square with struts. This has $4|V| - 12$ edges.



(a) Minimally infinitesimally rigid tensegrity framework in $\mathbb{R}^2$ with $4|V| - 12$ edges. This example is not generic.



(b) Slightly moving any of the central points, one of the struts attached to it becomes redundant.

Without any restrictions on p, we cannot hope for a significantly better upper bound.

Generic case conjecture

Conjecture

Let (T, p) be a minimally infinitesimally rigid generic realization of a simple tensegrity graph $T = (V, C \cup S)$ in $\mathbb{R}^d$ with $|V| \geq d + 2$. Then $|C \cup S| \leq (d+1)|V| - \binom{d+2}{2}$.

Conjecture ($d = 2$)

Let (T, p) be a minimally infinitesimally rigid generic realization of a simple tensegrity graph $T = (V, C \cup S)$ in $\mathbb{R}^2$ with $|V| \geq d + 2$. Then $|C \cup S| \leq 3|V| - 6$.

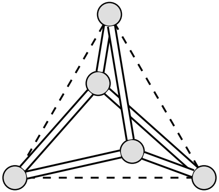
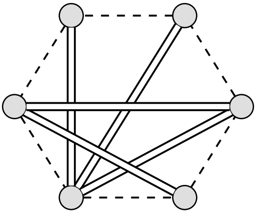


Figure: Minimally infinitesimally rigid generic tensegrity framework in $\mathbb{R}^2$ with $3|V| - 6$ edges.

Special case

We can prove the conjecture if we assume that all the vertices are in convex position and the subgraph (V, C) forms the boundary cycle of the convex hull.



Useful tools

The **rigidity matroid** of a framework (G, p) is defined on the edge set of G, where $F \subseteq E$ is independent if and only if the corresponding rows of the rigidity matrix $R(G, p)$ are linearly independent.

The **sparsity matroid** of a graph G is defined on the edge set of G, where the edge set of a subgraph H of G is independent if and only if H is sparse, i.e. for every vertex subset X (of at least 2 vertices) in H, the number of edges spanned by X is at most $2|X| - 3$.

If p is generic, then the rigidity matroid depends only on G and by Laman's theorem it is equivalent to the sparsity matroid of G.

Useful tools

Theorem (Roth, Whiteley)

Let (T, p) be a tensegrity framework in $\mathbb{R}^d$. Then (T, p) is infinitesimally rigid if and only if $(\overline{T}, p)$ is infinitesimally rigid and there exists a proper stress of (T, p).

Lemma (Roth, Whiteley)

Let (T, p) be a realization of the tensegrity graph $T = (V, C \cup S)$ in $\mathbb{R}^d$ and let $e \in C \cup S$. If there exists a stress of (T, p) with e in its support, then there exists a stress ω of (T, p) with e in its support and such that $\mathrm{rank} R_A(T, p) = |A|$ for every $A \subset \mathrm{supp}(\omega)$.

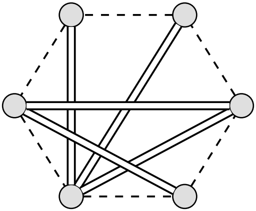
Lemma

Let (T, p) be a minimally infinitesimally rigid tensegrity framework in $\mathbb{R}^d$. Then every infinitesimally rigid subframework of (T, p) is minimally infinitesimally rigid.

Corollary

If (T, p) is minimally infinitesimally rigid then each edge of T is in a minimally infinitesimally rigid sub-tensegrity that forms a circuit in the rigidity matroid of T.

In our special case each of these rigid circuits must contain the subgraph (V, C) and by their minimality they are induced by their vertex sets. So T is a matroid circuit, therefore it has at least 4 vertices and exactly $2|V| - 2$ edges. Thus the conjecture holds.



Plans

Our goal is to prove the conjecture for 2-dimensions. Promising results:

- enough to prove it for tensegrities with connected rigidity matroids, which implies that there exists a matroid ear decomposition of the rigidity matroid
- if we managed to prove that there always exists a matroid ear decomposition of the rigidity matroid of T where there is no trivial ear then the conjecture would follow

Thank you for the attention!