Random matrices, perturbations and their applications in statistics

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Introduction, motivations

Singular vectors and singular values of matrices

Most important results, theorems

G Future plans, references

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• Importance of examining data and covariance.

• How can we describe matrices of data effectively?

• Perturbation problem: how can we model noisy observation.

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Singular vectors and values of matrices

- First singular value and vector of A.
- $\sigma_1 := \max_{|\nu|=1} |A\nu|$ and $\nu_1 := \operatorname*{argmax}_{|\nu|=1} |A\nu|$.
- By induction, let σ_i be the *i*-th singular value of matrix A (for i = 2...r) and let denote the *i*-th singular vector of matrix A by v_i, if

$$\sigma_i = \max_{v:|v|=1, v \perp v_1, \dots, v_{i-1}} |Av| \quad \text{and} \quad v_i = \operatorname*{argmax}_{|v|=1, v \perp v_1, v_2 \dots, v_{i-1}} |Av|.$$

• Features of singular vectors and values.

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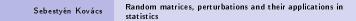
Interesting theorem (O'Rourke, Vu, Wang, 2016): If $E = [E]_{i,j}$ is a squared real symmetric matrix with independent entries and zero mean in and above the main diagonal and there exists a $K \ge 1$ with

$$P(E_{i,j} < K) = 1$$
 (for every i, j)

then for every normalized vectors u, v (|u| = |v| = 1) and every t > 0 we have

$$P((Eu)^T v \ge t) \le 2 \exp\left(-\frac{t^2}{K^2}\right)$$

The proof of this statement was not elaborated in the paper, I worked out the details and studied some generalizations.



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Bernoulli matrices

• The matrix E is called Bernoulli matrix if

$$E = [E]_{i,j}, \qquad P(E_{i,j} = 1) := P(E_{i,j} = -1) := 0.5$$

with independent coordinates.

• We have seen the main theorem of O'Rourke, Vu, Wan with Bernoulli matrix: if A is data matrix with (low) rank r and E is a random Bernoulli matrix, then for every $\varepsilon > 0$ there exist constants $C, \delta_0 > 0$ such that if

$$\delta \ge \delta_0$$
 and $\sigma_1 \ge \max\{n, \sqrt{n} \cdot \delta\}.$

then with a probability at least 1-arepsilon the inequality

$$\sin\left(<(v_1,v_1')\right) \leq C \cdot \frac{\sqrt{r}}{\delta}$$

fulfils.

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• Making simulations about perturbed random matrices.

• Understanding the perturbed random matrices and their statistical applications.

• Seeing the connections between these and Principal Component Analysis (PCA).

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Thank you for your attention!

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