Comparison of iterative methods for discretized nonsymmetric elliptic problems

Student: Lados Bálint István **Supervisor:** Karátson János

Motivation

A system of linear equations is often solved with an iterative method. A vast number of such methods have been proposed for nonsymmetric systems, however none of them can be considered the best. The performance of the most significant nonsymmetric matrix iterations has been studied in [1], where they give examples for cases when each iterative method outperforms the others.

The aim of this project is to investigate the special case when the nonsymmetric system comes from elliptic convection-diffusion PDEs with finite difference discretization, and examine how the results depend on the coefficients of the PDE.

Nonsymmetric iterative methods

In this project two commonly used iterative methods are being compared: CGN and GCR, more precisely their preconditioned version. CGN is the conjugate gradient method applied to the normal equation, while GCR (Generalized Conjugate Residual) is the mathematical equivalent of GMRES which is a Krylov subspace method that minimizes the residual norm, see [2].

The first part of the project was to create the preconditioned version of the algorithms. Preconditioning means that instead of directly solving the linear system of equations $Ax = b$, we rather consider the preconditioned system $B^{-1}Ax = B^{-1}b$ where we choose B in such a way that the iterative method converges more rapidly.

After the algorithms had been established, I created two MATLAB functions that execute the preconditioned CGN and GCR methods on the provided matrix (A) and vector (b), and return the number of iterations required for the B-norm of the residual to become sufficiently low. The tolerance level was set to 10^{-5}. The reason for this is that the GCR algorithm is sensitive to round-off errors when the matrix is ill-conditioned. Therefore, with double precision arithmetic and lower tolerance, the GCR method often did not converge and the comparison of the methods was not possible. For more details, see [3].

Elliptic convection-diffusion PDEs

Let us consider the following boundary value problem:

$$\begin{cases} Lu := -\mathrm{div}(p\nabla u) + \mathbf{w}\cdot\nabla u = f \\ u|_{\partial\Omega} = 0 \end{cases}$$

where $\Omega = [0,1]^2$, $p \in L^\infty(\Omega)$, $p(x) \geq m > 0$ (a.e. $x \in \Omega$), $\mathbf{w} \in C^1(\overline{\Omega}, \mathbb{R}^2)$ and $\mathrm{div}\mathbf{w} = 0$. These conditions guarantee that the PDE has a unique weak solution for any function $f \in L^2(\Omega)$.

The exact solution of the PDE can be approximated with the finite difference method which involves introducing a set of discrete grid points in Ω where the partial derivatives are approximated with an appropriate finite difference scheme. This discretization process results in a system of linear equations that can be solved by an iterative method. For this project, the preconditioner matrix was chosen to be the discretized form of $Su := -\text{div}(p\nabla u)$.

Comparison of iterative methods

The basis of comparison was the size of $\|\mathbf{w}\|_\infty$ and how it affects the number of iterations. First of all, let us consider the PDE with the following functions:

$$p(x,y) := 1 + \frac{x^2+y^2}{2}; \quad \mathbf{w}_k(x,y) := k \cdot \left(y - \frac{1}{2}, -x - \frac{1}{2}\right); \quad f \equiv 1.$$

The norm of the vector field was increased by choosing large positive numbers k in the definition of $\mathbf{w}_k$. After that, the number of iterations was measured for both methods with different grid densities and figure 1 was obtained. There are two important features that appear on these graphs: a finer grid results in a steeper curve, and the norm of the vector field highly influences the convergence of the methods. If we compare the two curves, we can see that CGN performs better for smaller norms, but after the intersection point of the curves, GCR takes over.

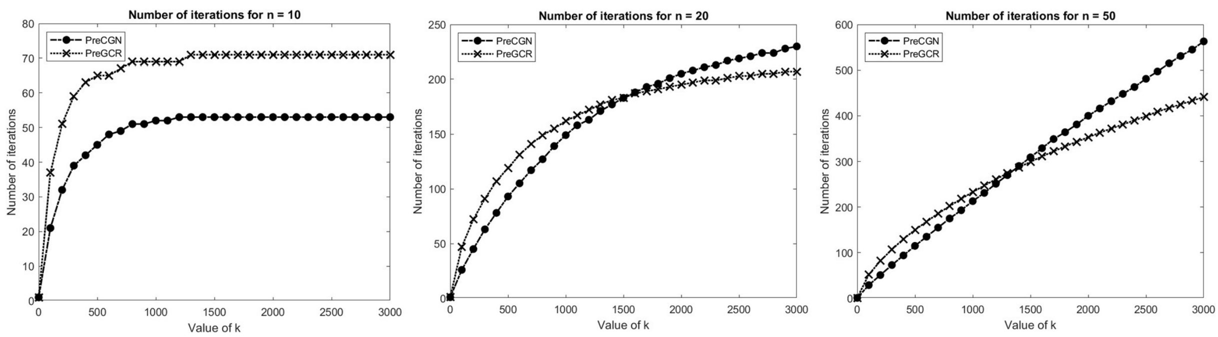

Figure 1: Number of iterations taken by the preconditioned CGN and GCR algorithms in the first example with respect to k and tolerance 10^{-5}. Each measurement was conducted with different grid densities ($n = 10,\ 20,\ 50$).

After testing with various vector fields and coefficients including exponential, trigonometric and rational functions, very similar results have been achieved. This suggests that the shape of the curves is less dependent on the actual coefficients, and it is rather characterised by the properties of the resulting matrix such as condition number and antisymmetry.

Conclusion

Regarding nonsymmetric elliptic problems, we saw that the performance of the preconditioned CGN and GCR methods depends on the coefficients of the PDE, therefore neither of them can be considered best in general. However, for dense grids we showed that CGN converges faster when the norm of the vector field in the nonsymmetric part of the PDE is smaller, while GCR converges faster when the norm is larger.

References

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