Comparison of iterative methods for discretized nonsymmetric elliptic problems

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Nonsymmetric elliptic PDE

Let us consider the following boundary value problem on $\Omega = [0,1]^2$

$$\begin{cases} Lu := -\operatorname{div}(p\nabla u) + \mathbf{w} \cdot \nabla u = f, \\ u_{|\partial\Omega} = 0. \end{cases}$$

If the functions satisfy these conditions:

$$\begin{cases} p \in L^\infty(\Omega),\ p(x) \geq m > 0 \text{ (a. e. } x \in \Omega); \\ \mathbf{w} \in C^1(\overline{\Omega}, \mathbb{R}^2),\ \operatorname{div} \mathbf{w} = 0, \end{cases}$$

then the PDE has a unique weak solution for any $f \in L^2(\Omega)$.

The exact solution of the PDE can be approximated with the finite difference method (FDM).

The FDM

Three steps of FDM:

- introduce a set of discrete grid points in Ω
- approximate the partial derivatives
- obtain a system of linear equations

Problem: How to solve the system of linear equations $Ax = b$, if A is a nonsymmetric matrix that comes from an elliptic PDE?

Solution: Use a preconditioned nonsymmetric matrix iteration.

Question:

- How do the coefficients of the PDE affect the convergence of the iterative methods?

Iterative methods

In this project two nonsymmetric iterative methods are being compared:

- **CGN:** the conjugate gradient method applied to the normal equation
- **GCR:** a Krylov subspace method that minimizes the residual norm

Preconditioning:

- Solve $B^{-1}Ax = B^{-1}b$.
- B is the discretized form of $-\operatorname{div}(p\nabla u)$.

Implementation: Matlab.

Model problems

Let us consider the PDEs with the following functions:

$$p(x,y) := 1+\frac{x^2+y^2}{2}; \quad \mathbf{w}_k(x,y) := k\cdot\left(y-\frac{1}{2}, -x-\frac{1}{2}\right); \quad f \equiv 1.$$

Comparison: Which iterative method performs better when $\|\mathbf{w}_k\|_\infty$ is small (k is small) and when it is large (k is large)?

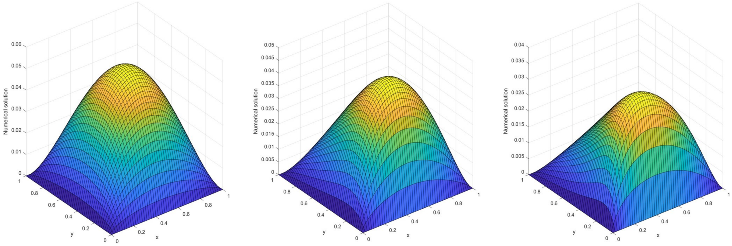

Figure: Numerical solution for $k = 1, 10, 20$.

Comparison of the iterative methods

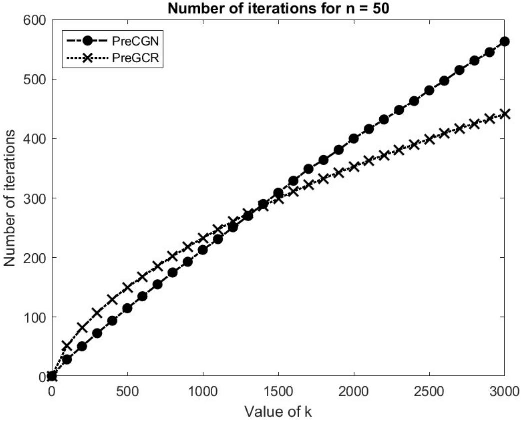

Figure: Number of iterations taken by the preconditioned CGN and GCR algorithms with respect to k, grid density $n = 50$ and tolerance 10^{-5}.

Results and future directions

Considering the studied model problems, we showed that:

- the **CGN** method converges faster when $\|\mathbf{w}_k\|_\infty$ is **small**;
- the **GCR** method converges faster when $\|\mathbf{w}_k\|_\infty$ is **large**.

Further questions to investigate:

- What determines the intersection point of the curves?
- How do other iterative methods relate to each other?
- What happens when we discretize the PDE with the finite element method (FEM) instead of the FDM?
- What is the effect of increasing the norm of other coefficients, such as function p?

References

Thank You for Your Attention!

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