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Free-rooted packings of arborescences

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1 Introduction

In this report we prove new results about packings of arborescences. An r-arborescence is a directed tree in which each node has an in-degree of 1 except the root note r, which has an in-degree of 0. A packing of subgraphs in a graph means a collection of subrgraphs, that are edge-disjoint. The most fundamental result of the study of packings of arborescences is the following result of Edmonds ([4]):

Theorem 1.1. *(Weak Edmonds Theorem [4]) Let $D = (V, A)$ a digraph and $r \in V$. There exists a packing of spanning r-arborescences in G if and only if*

$$\varrho_A(X) \geq k \text{ for all } \emptyset \neq X \subseteq V - r, \tag{1}$$

where $\varrho_A(X)$ denotes the in-degree of X.

This result has been generalized in multiple ways. Durand de Gevigney, Nguyen and Szigeti characterized the existence of matroid-based packings of arborescences in [3], Cs. Király, Szigeti, Tanigawa characterized the existence of matroid-based and matroid-restricted packings of arborescences in [7] and Bérczi and Frank characterized the existence of free-rooted packings of arborescences ([2]): packings, where the roots of the arborescences are not given (for further definitions see later sections).

This report generalizes results on free-rooted packings of arborescences. In section 4 we characterize the existence of free-rooted matroid-based and matroid-restricted packings of arboresences and prove some corollaries. In section 5 we extend a result of Szigeti ([8]) about free-rooted packings of arborescences in mixed graphs. Every proof in this report is original.

2 Definitions

Given a function $f : S \to \mathbb{R}$ and a finite set $Z \subset S$, let $\widetilde{f}(Z) := \sum_{s \in Z} f(s)$. Two subsets $X, Y \subseteq S$ are **intersecting**, if $X \cap Y \neq \emptyset$. A set function b on the ground set S is **subcardinal**, if $b(X) \leq |X|$ for all $X \subseteq S$, **submodular**, if

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y) \text{ for all } X, Y \subseteq S, \tag{2}$$

and **supermodular**, if

$$b(X) + b(Y) \leq b(X \cap Y) + b(X \cup Y) \text{ for all } X, Y \subseteq S. \tag{3}$$

A set function is **positively intersecting submodular** (**positively intersecting supermodular**), if (2) (respectively (3)) holds for intersecting subsets of S, for which $p(X) > 0$, $p(Y) > 0$.

Let $G = (S, T, E)$ be a bipartite graph and $p : T \to \mathbb{Z}$ a positively intersecting supermodular setfunction for which $p(\emptyset) = 0$ holds. Let $|\Gamma(Y)|$ be the set of neighbours of Y in G. We say that G covers p, if $\forall Y \subseteq T : p(Y) \leq |\Gamma(Y)|$. Given a matroid $M = (S, r)$, we say that G M-covers p, if $\forall Y \subseteq T : p(Y) \leq r(\Gamma(Y))$.

Let $D = (V + s, A)$ be a rooted digraph, where s is called the root. The in-degree of s is 0 and the outgoing edges are called **root-edges**. We call an s-rooted arborescence an **s-arborescence**. For $X, Z \subseteq V + s$, $B \subseteq A$ let $\partial_Z(X)$ denote the set of edges that go from $Z - X$ to X and let $\varrho_Z(X) = |\partial_Z(X)|$.

Let $M_1 = (\partial_s(V), r_1)$ be a matroid on the root-edges of D. We call a packing of s-arborescences $T_1, \dots, T_k$ **M_1-based**, if every T_i contains exactly one root-edge (e_i) and, for all vertices $v \in V$, $\{e_i : v \in V(T_i)\}$ is a basis of M_1. Let $M_2 = (A, r_2)$ be a matroid on the edges of D. We call a packing of s-arborescences **M_2-restricted** if the union of the edge sets of the arborescences in the packing is independent in M_2.

3 Background results

The following theorem is a stronger version of Theorem 1.

Theorem 3.1 (Strong Edmonds Theorem [4]). *Let $D = (V + s, A)$ be a rooted digraph, and let $\{B_1, \dots, B_k\}$ be a partition of its root-edges. There exists a packing of $T_1, \dots, T_k$ spanning s-arborescences, where the root-edges of T_i are in B_i for every $i = 1, \dots, k$ if and only if $\varrho_V(X) \geq |\{i \in \{1, \dots, k\} : B_i \cap \varrho_s(X) = \emptyset\}|$ for all $\emptyset \neq X \subseteq V$.*

In [1], Bérczi and Frank characterized the existence of a packing of spanning arborescences without specified root-sets, which we call a free-rooted packing.

Theorem 3.2. *(Bérczi, Frank [1]) Let $D = (V, A)$ be a digraph with n nodes and let $\mu_1, \dots, \mu_k$ positive integers. The following statements are equivalent:*

(A) *There exists in D a packing of k edge-disjoint spanning arborescences $B_1, \dots, B_k$, for which $|B_i| = \mu_i$ for all $i = 1, \dots, k$.*

(B1) *For every subpartition $\{V_1, \dots, V_q\}$ of V:*

$$\sum_{j=1}^{k} \max\{0, q - (n - \mu_j)\} \leq \sum_{i=1}^{q} \varrho(V_i) \tag{4}$$

(B2) *Let $[k] = \{1, 2, \dots, k\}$. For every subpartition $\{V_1, \dots, V_q\}$ of V and for all $X \subseteq [k]$:*

$$|[k] - X|q - \sum_{j \in [k]-X} n - \mu_j \leq \sum_{i=1}^{q} \varrho(V_i) \tag{5}$$

Their proof relies on the following theorem:

Theorem 3.3. *(Bérczi, Frank [1]) Let m_S be a degree-specification on S for which $\widetilde{m}_S(S) = \gamma$. Let p_T be a positively intersecting supermodular function on T with $p_T(\emptyset) = 0$. Suppose that*

$$m_S(s) \leq |T| \ \forall s \in S. \tag{6}$$

The following statements are equivalent:

(A) *There exists a simple bipartite graph $G = (S, T, E)$, which covers p_T and fits the degree-specification m_S*

(B1) *For every subpartition $\{T_1, \dots, T_q\}$ of T and $X \subseteq S$:*

$$\widetilde{m}_S(X) + \sum_{i=1}^{q} p_T(T_i) - q|X| \leq \gamma \tag{7}$$

(B2) *For every subpartition $\{T_1, \dots, T_q\}$ of T:*

$$\sum_{i=1}^{q} p_T(T_i) \leq \sum_{s \in S} \min\{m_S(s), q\} \tag{8}$$

In [2], a generalization of Theorem 3.3 is provided:

Theorem 3.4. *(Bérczi, Frank [2]) We are given a matroid $M = (S, r)$, a positively intersecting supermodular function p_T on T and a degree-specification m_S on S, for which $\widetilde{m}_S(S) = \gamma$. There is a simple bigraph $G = (S, T, E)$, which M-covers p_T and fits m_S if and only if*

$$m_S(s) \leq |T| \ \forall s \in S \tag{9}$$

and for every subpartition $\{T_1, \dots, T_q\}$ of T and $X \subseteq S$:

$$\widetilde{m}_S(X) + \sum_{i=1}^{q} p_T(T_i) - qr(X) \leq \gamma \tag{10}$$

In [3], Durand de Gevigney, Nguyen and Szigeti characterized the existence of so called matroid-based packings of arborescences:

Theorem 3.5. *(Durand de Gevigney, Nguyen, Szigeti [3]) We are given a graph $D = (V + s, A)$ and a matroid $M = (\partial_s(V), r)$. There is an M-based packing of s-arborescences in D if and only if*

$$\varrho_V(X) \geq r(M) - r(\partial_s(X)) \tag{11}$$

Furthermore, if we want the S to be the root set of arborescences, then the following must also hold

$$\partial_s(v) \text{ is independent in } M \text{ for every } v \in V. \tag{12}$$

The following theorem characterizes the existence of matroid-based and matroid-restricted packings of s-arborescence, which is a generalization of Theorem 3.5.

Theorem 3.6. *(Cs. Király, Szigeti, Tanigawa [7]) We are given a graph $D = (V + s, A)$, a matroid $M_1 = (\partial_s(V), r_1)$ with a rank function r_1, a matroid M_2 on A, which is the direct sum of the matroids $M_v = (\partial(v), r_v)$. There exist in D an M_1-based M_2-restricted packing of s-arborescences if and only if*

$$r_1(F) + r_2(\partial(X) - F) \geq r_1(\partial_s(V)) \tag{13}$$

for all $\emptyset \neq X \subseteq V$ and $F \subseteq \partial_s(X)$. If on the neighbouring edges of s $M_2 = M_2|_{\partial_s(V)} \oplus M_2|_{E(V)}$ and $M_2|_{\partial_s(V)}$ is the free matroid, then the condition is the following:

$$r_1(\partial_s(X)) + r_2(\partial(X) - \partial_s(X)) \geq r_1(\partial_s(V)) \tag{14}$$

for all $\emptyset \neq X \subseteq V$.

4 Free-rooted packings of arborescences with matroid constraints

In this section we characterize the existence of free-rooted matroid-based and matroid-restricted packings of arborescences, give two characterizations of the existence of free-rooted matroid-based packings of arborescences with an in-degree prescription and provide a new characterization for the existence of a free-rooted arborescence packing with an in-degree prescription.

Using Theorem 3.6 and Theorem 3.4, we can characterize the existence of a free-rooted matroid-based and matroid restricted packing of arborescences.

Theorem 4.1. *Let $D = (V, A)$ be a digraph, let $M_1 = (S, r_1)$ be a matroid with rank function r_1 and rank k and let M_2 be a matroid on A which is the direct sum of the matroids $M_v = (\partial(v), r_v)$. Let s be a node not in V. The following statements are equivalent:*

(A) *We can add new possibly parallel arcs from s to some of the nodes of V and we can assigne the elements of S to the new edges such that there exists an M_1-based M_2'-restricted packing of s-arborescences, where M_2' the direct sum of the free matroid on the new edges and M_2.*

(B) *For every subpartition $\{V_1, \dots, V_q\}$ of V and $X \subseteq S$:*

$$(k - r_1(X))q - |S - X| \leq \sum_{i=1}^{q} r_2(\partial(V_i)) \tag{15}$$

Proof. Necessity. Suppose that such a packing exists. Then at most $r_2(\partial(Y))$ and at least $k - \partial_s(Y)$ edges of the packing enter a set $Y \subset V$, thus

$$\sum_{i=1}^{q} (k - r_1(\partial_s(V_i))) \leq \sum_{i=1}^{q} r_2(\partial(V_i))$$

Using the properties of the rank function we can show that, for every $X \subseteq S$

$$\sum_{i=1}^{q} r_1(\partial_s(V_i)) \le \sum_{i=1}^{q} r_1(\partial_s(V_i)\cap X) + r_1(\partial_s(V_i) - X)) \le q r_1(X) + |S - X|.$$

Hence

$$\sum_{i=1}^{q} r_2(\partial(V_i)) \ge \sum_{i=1}^{q} (k - r_1(\partial_s(V_i))) \ge qk - q r_1(X) - |S - X|.$$

Sufficiency. Let $m_S : S \to \mathbb{Z}_+$ be 1 for every element of S. Let $T := V$ and let us define the following intersecting supermodular function on T:

$$p_T(Y) = \begin{cases} k - r_2(\partial(Y)) & \varnothing \subset Y \subseteq T, \\ 0 & Y = \varnothing. \end{cases}$$

From the conditions of the theorem:

$$(k - r_1(X))q - \sum_{i=1}^{q} r_2(\partial(V_i)) \le |S - X| = \widetilde{m}_S(S - X) = \widetilde{m}_S(S) - \widetilde{m}_S(X)$$

$$-r_1(X)q + \sum_{i=1}^{q} (k - r_2(\partial(V_i))) \le \widetilde{m}_S(S) - \widetilde{m}_S(X)$$

$$\sum_{i=1}^{q} p_T(V_i) + \widetilde{m}_S(X) - r_1(X)q \le \widetilde{m}_S(S)$$

This is the condition of Theorem 3.4, therefore there exists a simple bipartite graph $G = (S, V, E)$, which covers p_T and satisfies m_S, that is $r_1(\Gamma(Y)) \ge k - \varrho(Y)\ \forall Y \subset V$. Direct the edges of G from S to T, add the edges of D in T and contract the nodes of S into a new node s. $\Gamma(Y) = \partial_s(Y)$ holds therefore, since G covers p_T, $r_1(\partial_s(Y)) \ge k - r_2(\partial(Y))$ holds, which is the condition of Theorem 3.6 with matroids M_1 and M_2', which means that there exists an M_1-based M_2-restricted packing of s-arborescences. □

Using the previous theorem, we can characterize the existence of a free-rooted matroid-based packing of arborescences with an in-degree prescription:

Collorary 4.1. *Let $M = (S, r)$ be a matroid with rank function r, let $D = (V, A)$ be a digraph with n nodes and let $m_{in} : V \to \mathbb{Z}^+$ be an in-degree prescription for which $0 \le m_{in}(v) \le \varrho_D(v)$, $m_{in}(V) \le r(M)$ for all $v \in V$ and $\widetilde{m}_{in}(V) = |V| r(M) - |S|$ holds. Let s be a node not in V. The following statements are equivalent:*

- ***(A)*** *We can add new arcs from s to some of the nodes of V and we can assigne the elements of S to the new edges such that there exists an M-based s-arborescence packing and if the edge set of the packing whitout the root edges is F, then $\varrho_F(v) = m_{in}(v)$ holds for every $v \in V$.*
- ***(B)*** *For all $X \subseteq S$ and subpartition $\{V_1, \ldots, V_q\}$ of V:*

$$(r(M) - r(X))q - |S - X| \le \sum_{i=1}^{q} \sum_{v \in V_i} \min\{m_{in}(v), |\partial(v) \cap \partial(V_i)|\} \qquad (16)$$

Proof. Let $M_1 := M$ and $\forall v \in V$ let M_v be the uniform matroid on $\partial(v)$ with rank $m_{in}(v)$. Let M_2 be the direkt sum of the matroids M_v. Then

$$r_2(\partial(V_i)) = \sum_{v \in V_i} \min\{m_{in}(v), |\partial(v) \cap \partial(V_i)|\},$$

so

$$\sum_{i=1}^{q} r_2(\partial(V_i)) = \sum_{v \in \bigcup_{i=1} V_i} \min\{m_{in}(v), |\partial(v) \cap \partial(V_i)|\}.$$

So (16) is the same as the condition of Theorem 4.1, so there exists a M_1-based M_2-restricted packing. This means that at most $m_{in}(v)$ arborescence enters every node v. Since $\widetilde{m}_{in}(V) = |V|r(M) - |S|$ and the right side is the number of edges in an M-based restriction, exactly $m_{in}(v)$ edge enters every node. □

Using Collorary 4.1 we can prove a new characterization for the existence of a free-rooted packing of arborescences with an in-degree prescription.

Collorary 4.2. *let $D = (V, A)$ be a digraph with n nodes and let $m_{in} : V \to \mathbb{Z}^+$ be an in-degree prescription for which $0 \leq m_{in}(v) \leq \varrho_D(v)$ and $m_{in}(V) \leq k$ for all $v \in V$. Let $\mu_1, \dots, \mu_k$ be kpositive integers, for which $\sum_{i=1}^k \mu_i = \widetilde{m}_{in}(V)$. The following statements are equivalent:*

(A) *There exist in D a packing of spanning arborescenses $B_1, \dots, B_k$ k, for which $|B_i| = \mu_i$ and if $\bigcup_{i=1}^k B_i = F$, than $v \in V : \varrho_F(V) = m_{in}(v)$.*

(B) *For every subpartition $\{V_1, \dots, V_q\}$ of V:*

$$\sum_{i=1}^k \max\{0, q - (n - \mu_i)\} \leq \sum_{v \in \bigcup_{i=1}^q V_i} \min\{m_{in}(v), |\partial(v) \cap \partial(V_i)|\} \tag{17}$$

Proof. Let $n - \mu_j := m_j$. This is the number of roots for a spanning arborescence with μ_j edges.

Let M_1 be a partition matroid with k classes, where the size of the i. class is m_i and the bound is 1 for every class. (Let M_2 be the same matroid as in the previous proof.) According to Collorary 4.1. $(A) \Leftrightarrow (r(M) - r(X))q - |S - X| \leq \sum_{v \in \bigcup_{i=1}^q V_i} \min\{m_{in}(v), |\partial(v) \cap \partial(V_i)|\}$. We can assume that the set X contains either the entire partition class or it is disjoint from it. This is because if it intersects a class, then if we add the elements from the class that are not contained in X, then the left side increase and the right side stays the same. So if $I = \{1, \dots, k\}$, then

$$(A) \Leftrightarrow (k - |X|)q - \widetilde{m}(S - X) \leq \sum_{v \in \bigcup_{i=1}^q V_i} \min\{m_{in}(v), |\partial(v) \cap \partial(V_i)|\} \forall X \subseteq I$$

The left side is maximized by $X = \{i \in I : m(i) > q\}$ and for this set $(k - |X|)q - \widetilde{m}(S - X) = \sum_{i=1}^k \max\{0, q - (n - \mu_i)\}$ holds. □

In [1], Bérczi and Frank provide a different characterization for the same problem with the following condition:

For all $Y \subseteq V$ and subpartition $\{V_1, \dots, V_q\}$ of $V - Y$:

$$\sum_{i=1}^k \max\{0, q + |Y| - (n - \mu_i)\} \leq \widetilde{m}_{in}(Y) + \sum_{i=1}^q \varrho_D(V_i) \tag{18}$$

This result follows from the following theorem, which gives a different characterization of the existence of a free-rooted matroid-based packing of arborescences with an in-degree prescription with a seemingly weaker condition. The proof is based on Theorem 3.5 and Theorem 3.4.

Theorem 4.2. *Let $M = (S, r)$ be a matroid with rank function r, let $D = (V, A)$ be a digraph with n nodes and let $m_{in} : V \to \mathbb{Z}^+$ be an in-degree prescription for which $0 \leq m_{in}(v) \leq \varrho_D(v)$, $m_{in}(V) \leq r(M)$ for all $v \in V$ and $\widetilde{m}_{in}(V) = |V|r(M) - |S|$ holds. Let s be a node not in V. The following statements are equivalent:*

(A) *We can add new arcs from s to some of the nodes of V and we can assign the elements of S to the new edges such that there exists an M-based s-arborescence packing and if the edge set of the packing whitout the root edges is F, then $\varrho_F(v) = m_{in}(v)$ holds for every $v \in V$.*

(B) *For all $Y \subseteq V$, subpartition $\{V_1, \dots, V_q\}$ of $V - Y$ and $X \subseteq S$:*

$$(|Y| + q)(r(M) - r(X)) - |S - X| \leq \widetilde{m}_{in}(Y) + \sum_{i=1}^q \varrho_D(V_i) \tag{19}$$

Furthermore, (16) implies (19).

Proof. First we will prove that (16) implies (19). By Theorem 4.1, this implies the neccesity of (19).

Let us suppose that (16) holds and we are given a set $Y \in V$ and a subpartition $\mathcal{P} = \{V_1, \dots, V_q\}$ of $V-Y$. Let us define the following partiton of V: $\mathcal{P}' = \mathcal{P} \cup \bigcup_{v \in Y}\{v\}$. Then $|\mathcal{P}'| = q+|Y|$, $m_{in}(Y) = \sum_{v\in Y} \min\{m_{in}(v), |\partial(v) \cap \partial(V_i)|\}$ és $\sum_{v \in \bigcup_{i=1}^{q} V_i} \min\{m_{in}(v), |\partial(v) \cap \partial(V_i)|\} \le \sum_{i=1}^{q} \varrho_D(V_i)$, so (19) holds.

Necessity: Let $m_S : S \to \mathbb{Z}_+$ be 1 for every element of S. Let $T := V$ and let us define the following set function on T:

$$p_T(Y) = \begin{cases} r(M) - \varrho_D(Y) & Y \subseteq T, |Y| \ge 2, \\ r(M) - m_{in}(v) & Y = \{v\}, v \in V \\ 0 & Y = \varnothing. \end{cases}$$

Since $m_{in}(v) \le \varrho_D(v)$, $k - m_{in}(v) \ge k - \varrho_D(v)$ so p_T is intersecting supermodular.

Let $\mathcal{T} = \{V_1, \dots, V_q, \dots, V_{q'}\}$ be a subpartition of T, where the last $q'-q$ class consist of only 1 element. Let $\mathcal{P} = \{V_1, \dots, V_q\}$ and $Y = V_{q+1} \cup \dots \cup V_{q'}$.

From the definition of p_T:

$$\sum_{i=1}^{q'} p_T(V_i) = \sum_{i=1}^{q} [r(M) - \varrho_D(V_i)] + \sum_{i=q+1}^{q'} [r(M) - \widetilde{m}_{in}(V_i)] = (|Y|+q)r(M) - \sum_{i=1}^{q} \varrho_D(V_i) - \widetilde{m}_{in}(Y)$$

If we apply the condition of Theorem 3.4 to $\mathcal{T}$ and a set $X \subset S$ and we use the previous equation for $\sum_{i=1}^{q'} p_T(V_i)$, we get the following:

$$\widetilde{m}_S(X) + (|Y|+q)r(M) - \sum_{i=1}^{q} \varrho_D(V_i) - \widetilde{m}_{in}(Y) - q'r(X) \le \widetilde{m}_S(S)$$

If we reorder the terms and use that $q' = |Y|+q$ and $\widetilde{m}_S(S) - \widetilde{m}_S(X) = |S-X|$ we get the condition in (19). So there exists a simple bipartite graph $G = (S, V, E)$, which covers p_T-t and satisfies the degree prescription. From this we get that for every $v \in V$-re $r(M) - m_{in}(v) \le r(\Gamma_G(v)) \le d_G(v)$ (where $\Gamma_G(v)$ is the set of neighbours of v in G), so

$$\sum_{v\in V} [r(M) - m_{in}(v)] \le \sum_{v \in V} d_G(v) = \sum_{s \in S} d_G(s) = |S|$$

Since $\widetilde{m}_{in}(V) = |V|r(M) - |S|$, the left hand side of the previous equation is $|S|$, so we have equality everywhere, so we get $d_G(v) = r(\Gamma_G(v)) = r(M) - m_{in}(v)$ $\forall v \in V$.

Since $\forall Y \subset V$-re $r(M) - \varrho_D(Y) \le r(\Gamma_G(Y))$ also holds, if we contract S and direct its outgoing edges similarly to the ending of the previous proof, then the condition of Theorem 3.5 holds for the graph, so there exists an M-based s-arborescence packing. Since $r(\Gamma_G(v)) = r(M) - m_{in}(v)$, at least $m_{in}(v)$ arborescence enters v with a non-rootedge. We can assprimeume that exactly $r(\Gamma_G(v))$ root-edge is in the packing (becouse otherwise we can exchange certain edges of the arborescences), so there exists a packing wich enters v with exactly $m_{in}(v)$ edges.

□

5 Free-rooted packings of arborescences in mixed graphs

A mixed graph is a graph which has both undirected and directed edges (arcs). In this section we prove a generalization of a result on free-rooted packings of arborescences in mixed graphs by Szigeti ([8]).

Let $F = (V, E \cup A)$ be a mixed graph where E is the set of undirected edges and A is the set of arcs. For $B \subseteq E \cup A$, let $\boldsymbol{\partial^B(X)}$ be the set of edges in B entering X. Let $\boldsymbol{\varrho^B(X)}= |\partial^B(X)|$. Orienting an edge we replace it with an arc. For $\overrightarrow{Z} \subseteq A$, Z denotes the underlying undirected edges of $\overrightarrow{Z}$. For $Z \subseteq E$ and $X \subseteq V$ the set of vertices covered by Z is denoted by $\boldsymbol{V(Z)}$ and the set of edges in Z that are induced by X is denoted by $\boldsymbol{Z(X)}$. A mixed r-arborescence is a mixed graph that can be oriented to be an r-arborescence. For a family of sets $\mathcal{P}$ on V and $B \subseteq A \cup E$ let $\boldsymbol{\partial_B(\mathcal{P})}$ be the set of (directed and undirected) edges in B, that enter a member of $\mathcal{P}$ and $\boldsymbol{\varrho_B(\mathcal{P})}$:$= |\partial_B(\mathcal{P})|$. If $f, v : V \to \mathbb{Z}_+$, then we call a packing of arborescences $\boldsymbol{(f,g)}$**-bounded**, if for each $v \in V$, the number of v-arborescences in the packing is between $f(v)$ and $g(v)$. For $k, l, l' \in \mathbb{Z}_+ - \{0\}$ a packing of arborescences is $\boldsymbol{(l,l')}$**-bounded** if the number of arborescences in the packing is between l and l', and $\boldsymbol{k}$**-regular**, if each vertex is in exactly k arborescences in the packing. We call a packing of mixed arborescences (f,g)-bounded/(l,l')/k-regular, if we can orient the undirected edges such that we get an (f,g)-bounded/(l,l')/k-regular packing of arborescences.

For a graph $G = (V, E)$, let $\mathbf{M_G}$ be the graphic matroid of G, and let $\mathbf{M_G^k}$ be the k-graphic matroid of G, that is the k-sum of M_G, which is a matroid on V, where a set is independent if and only if it can be partitioned into k independent sets of M_G. Let $F = (V, E \cup A)$ be a mixed graph. For a partition $\mathcal{P}$ of V, let $\boldsymbol{A(\mathcal{P})}$ and $\boldsymbol{E(\mathcal{P})}$ be the set of directed and undirected edges entering a member of $\mathcal{P}$. Let $\mathbf{G_F} = (V, E \cup E_A)$ be the underlying undirected graph of F, and $\mathbf{D_F} = (V, A_E \cup A)$ the directed extension of F, where $A_E = \bigcup_{e \in E} A_e$, and if $e = uv$, $A_e = \{\overrightarrow{uv}, \overrightarrow{vu}\}$ ($\overrightarrow{uv}$ is an arc from u to v). The extended k-hipergraphic matroid $\mathbf{M_F^k}$ of F is a matroid on $A \cup A_E$, which we get from $M^k_{G_F}$ by replacing each edge $e \in E$ with two paralell copies of itself, associating these edges to the corresponding edges in A_E, and associating the edges of E_A with the corresponding arcs of A. It is shown in [6], that the rank function of M_F^k is the following ($Z \subseteq A \cup A_E$):

$$r_{M_F^k}(Z) = \min\{|Z \cap A(\mathcal{P})| + |\{e \in E(\mathcal{P}) : \ Z \cap A_e \neq \emptyset\}| + k(|V| - |\mathcal{P}|) : \mathcal{P} \textit{ is a partition of } V\} \quad (20)$$

Let p and b be two set functions on S. For a vector $x \in \mathbb{R}^S$ and $Z \subseteq S$, let $\widetilde{x}(Z) := \sum_{s \in S} x_s$. The polyhedron $\boldsymbol{Q(p,b)}$= $\{x \in \mathbb{R}^S : p(Z) \leq \widetilde{x}(Z) \leq b(Z) \ \forall Z \subseteq S\}$ is called a generalized-polymatroid or g-polymatroid if p and b have the following properties: $p(\emptyset) = b(\emptyset)$, p is supermodular, b is submodular and $b(X) - b(Y) \geq b(X - Y) - p(Y - X)$ for all $X, Y \subseteq S$. The Minkowski sum of the n g-polymatroids $Q(p_i, b_i)$ is denoted by $\sum_1^n Q(p_i, b_i)$. For $\alpha, \beta \in \mathbb{R}$, the polyhedron $\mathbf{K}\boldsymbol{(\alpha,\beta)}$= $\{x \in \mathbb{R}^S : \alpha \leq \widetilde{x}(S) \leq \beta\}$ is called a plank. We will use the following results on g-polymatroids:

Theorem 5.1 (Frank [5]). *1. Let $Q(p,b)$ be a g-polymatroid, $K(\alpha,\beta)$ a plank and $M = Q(p,b) \cap K(\alpha,\beta)$.*

(i) $M \neq \emptyset$ if and only if $p \leq b$, $\alpha \leq \beta$, $\beta \geq p(S)$ and $\alpha \leq b(S)$.

(ii) M is a g-polimatroid.

(iii) If $M \neq \emptyset$, then $M = Q(p^\alpha_\beta, q^\alpha_\beta)$ with

$$p^\alpha_\beta(Z) = \max\{p(Z), \alpha - b(S - Z)\}$$
$$b^\alpha_\beta(Z) = \min\{b(Z), \beta - p(S - Z)\}$$

2. Let $Q(p_1, b_1)$ and $Q(p_2, b_2)$ be two non-empty g-polymatroids and $M = Q(p_1, b_1) \cap Q(p_2, b_2)$.

(i) $M \neq \emptyset$ if and only if $p_1 \leq b_2$ and $p_2 \leq b_1$.

(ii) If p_1, b_1, p_2, b_2 are integral and $M \neq \emptyset$, then M contains an integral element.

3. Let $Q(p_i, b_i)$ be n nonempty g-polimatroids. Then $\sum_1^n Q(p_i, b_i) = Q(\sum_1^n p_i, \sum_1^n b_i)$.

Theorem 5.2. *(Szigeti [8]) Let $F = (V, E \cup A)$ be a mixed graph, $f, g : V \to \mathbb{Z}_+$ functions and $k, l, l' \in \mathbb{Z}_+ - \{0\}$. There exists an (f,g)-bounded k-regular (l,l')-limited packing of arborescences in F if and only if $g_k(v) \geq f(v)$ for every $v \in V$, $\min\{\widetilde{g}_k(V), l'\} \geq l$ and*

$$\varrho_{A \cup E}(\mathcal{P}) \geq k|\mathcal{P}| - \min\{l' - f(\overline{\cup \mathcal{P}}), \widetilde{g}_k(\cup \mathcal{P})\} \ \textit{for every subpartition } \mathcal{P} \textit{ of } V \quad (21)$$

Theorem 5.3. *(Szigeti [8]) Let $F = (V, E \cup A)$ be a mixed graph, $f, g : V \to \mathbb{Z}_+$ functions and $k, l, l' \in \mathbb{Z}_+ - \{0\}$. Let $M_v := (\partial^{A \cup A_e}(v), r_v)$ be the free matroid for every $v \in V$, and let $M := \bigoplus_{v \in V} M_v$ with a rankfunction r. Let M_F^k be the extended k-graphic matroid of F on $A \cup A_e$. Let $T := (\sum_{v \in V}((Q(0, r_v)) \cap K(k - g_k(v), k - f(v))) \cap K(k|V| - l', k|V| - l) \cap Q(0, r_{M_F^k}))$.*

(A) *The characteristic vectors of the edge sets of (f,g)-bounded k-regular (l,l')-limited M-restricted packings of arborescences in orientations of F are exactly the integer points of T.*

(B) *$T \neq \emptyset$ if and only if $g_k(v) \geq f(v)$ for every $v \in V$, $\min\{\widetilde{g}_k(V), l'\} \geq l$ and for every $Z \subseteq A \cup A_E$,*

$$\sum_{v \in V} \max\{0, k - g_k(v) - r_v(\partial^Z(v))\} \leq r_{M_F^k}(\overline{Z}) \tag{22}$$

$$k|V| - l' - \sum_{v \in V} \min\{r_v(\partial^Z(v)), k - f(v)\} \leq r_{M_F^k}(\overline{Z}) \tag{23}$$

(C) *(25) and (26) are equivalent to (24).*

Theorem 5.4. *Let $F = (V, E \cup A)$ be a mixed graph, $f, g : V \to \mathbb{Z}_+$ functions and $k, l, l' \in \mathbb{Z}_+ - \{0\}$. Let $M_v := (\partial^{A \cup A_e}(v), r_v)$ be a matroid for every $v \in V$, and let $M := \bigoplus_{v \in V} M_v$ with a rankfunction r. There exists an (f, g)-bounded k-regular (l, l')-limited M-restricted packing of arborescences in F if and only if $g_k(v) \geq f(v)$ for every $v \in V$, $\min\{\widetilde{g}_k(V), l'\} \geq l$ and*

$$R(\mathcal{P}) \geq k|\mathcal{P}| - \min\{l' - f(\overline{\cup\mathcal{P}}), \widetilde{g}_k(\cup\mathcal{P})\} \text{ for every subpartition } \mathcal{P} \text{ of } V \tag{24}$$

where $R(\mathcal{P}) = \max\{r(\overrightarrow{\partial_{A \cup E}(\mathcal{P})})\}$, where $\overrightarrow{\partial_{A \cup E}(\mathcal{P})}$ is an orientation of $\partial_{A \cup E}(\mathcal{P})$ in which every undirected edge is oriented in such a way that it enters a member of $\mathcal{P}$.

The proof of this theorem relies on the following theorem:

Theorem 5.5. *Let $F = (V, E \cup A)$ be a mixed graph, $f, g : V \to \mathbb{Z}_+$ functions and $k, l, l' \in \mathbb{Z}_+ - \{0\}$. Let $M_v := (\partial^{A \cup A_e}(v), r_v)$ be a matroid for every $v \in V$, and let $M := \bigoplus_{v \in V} M_v$ with a rankfunction r. Let M_F^k be the extended k-graphic matroid of F on $A \cup A_e$. Let $T := (\sum_{v \in V}((Q(0, r_v)) \cap K(k - g_k(v), k - f(v))) \cap K(k|V| - l', k|V| - l) \cap Q(0, r_{M_F^k}))$.*

(A) *The characteristic vectors of the edge sets of (f, g)-bounded k-regular (l, l')-limited M-restricted packings of arborescences in orientations of F are exactly the integer points of T.*

(B) *$T \neq \emptyset$ if and only if $g_k(v) \geq f(v)$ for every $v \in V$, $\min\{\widetilde{g}_k(V), l'\} \geq l$ and for every $Z \subseteq A \cup A_E$,*

$$\sum_{v \in V} \max\{0, k - g_k(v) - r_v(\partial^Z(v))\} \leq r_{M_F^k}(\overline{Z}) \tag{25}$$

$$k|V| - l' - \sum_{v \in V} \min\{r_v(\partial^Z(v)), k - f(v)\} \leq r_{M_F^k}(\overline{Z}) \tag{26}$$

(C) *(24) implies (25) and (26).*

Proof. **(A)**

By Theorem 5.3/(A), the integer points of T are characteristic vectors of the edge sets of (f, g)-bounded k-regular (l, l')-limited packings of arborescences in orientations of F. Since an integer point of T is also in $\sum_{v \in V} Q(0, r_v)$, the corresponding packing is also M-based.

By the other direction of Theorem 5.3/(A), since the characteristic vector of an M-based packing must be in $\sum_{v \in V} Q(0, r_v)$, the integer points of T are exactly the characteristic vectors of the edge sets of the required packings.

(B)

By Theorem 5.1.1 $Q(0, r_v)) \cap K(k - g_k(v), k - f(v))$ is non empty if and only if $k - g_k(v) \leq k - f(v)$ (which is equivalent to $g_k(v) \geq f(v)$), $k - f(v) \geq 0$ (which is true because $0 \leq k - g_k(v) \leq k - f(v)$) and $k - g_k(v) \leq r_v(\partial^{A \cup A_E}(v))$ (which we will see later).

If $Q(0, r_v)) \cap K(k - g_k(v), k - f(v)) \neq \emptyset$ then it is equal to $Q(p_v, b_v)$, where by Theorem 5.1.1/(iii), for $Z \subseteq A \cup A_E$ and $Z_v = Z \cap \partial^{A \cup A_E}(v)$,

$$p_v(Z_v) = \max\{0, k - g_k(v) - r(\partial^{\overline{Z}_v}(v))\}$$

$$b_v(Z_v) = \min\{r(\partial^{Z_v}(v), k - f(v)\}$$

By Theorem 5.1.3, $\sum_{v \in V} Q(p_v, b_v) = Q(p_\Sigma, b_\Sigma)$, where $p_\Sigma = \sum_{v \in V} p_v$, $b_\Sigma = \sum_{v \in V} b_v$.

By Theorem 5.1.1, $Q(0, r_{M_F^k}) \cap K(k|V| - l', k|V| - l) \neq \emptyset$ if and only if $k|V| - l' \leq k|V| - l$ (which is equivalent to the second half of $\min\{\widetilde{g}_k(V), l'\} \geq l$), $k|V| - l \geq 0$ (which follows from $k|V| - l \geq \widetilde{g}_k(V) - l \geq 0$) and $k|V| - l' \leq r_{M_F^k}(A \cup A_e)$, which is (26) for $Z = A \cup A_E$. Then $Q(0, r_{M_F^k}) \cap K(k|V| - l', k|V| - l) = Q(p, b)$, where $p(Z) = \max\{0, k|V| - l' - r_{M_F^k}(\overline{Z})\}$ and $b(Z) = \min\{r_{M_F^k}(Z), k|V| - l\}$.

By Theorem 5.1.2, $Q(p, b) \cap Q(p_\Sigma, b_\Sigma) \neq \emptyset$ if and only if $p_\Sigma \leq b$ and $p \leq b_\Sigma$, that is

$$\sum_{v \in V} \max\{0, k - g_k(v) - r_v(\partial^{\overline{Z}_v}(v))\} \leq \min\{r_{M_F^k}(Z), k|V| - l\}$$

and

$$\max\{0, k|V| - l' - r_{M_F^k}(\overline{Z})\} \leq \sum_{v \in V} \min\{r_v(\partial^{Z_v}(v)), k - f(v)\}$$

The first inequality is equivalent to (25) by the fact, that $\max\{0, k|V| - l' - r_{M_F^k}(\overline{Z})\} \leq \sum_{v \in V} k - g_k(v) \leq k|V| - l$ (here we use that $\min\{\widetilde{g}_k(V), l'\} \geq l$). The first part of the second inequality is equivalent to $k - f(v) \geq k - g_k(v) \geq 0$ ($r_v \geq 0$ always holds), and the second part is equivalent to (26).

Finally, $k - g_k(v) \leq r_v(\partial^{A \cup A_E}(v))$ follows from $p_\Sigma(\emptyset) \leq b(\emptyset)$ and the proof is complete.

(C)

Note, that (25) is equivalent to

$$k|V| - g_k(V) - \sum_{v \in V} \min\{r_v(\partial^Z(v)), k - g_k(V)\} \leq r_{M_F^k}(\overline{Z}). \tag{27}$$

Let $Z \subseteq A \cup A_E$. By (20), there exists a partition $\mathcal{P}$ of V such that for $K = \{e \in E(\mathcal{P}) : \overline{Z} \cap A_e \neq \emptyset\}$:

$$r_{M_F^k}(\overline{Z}) = |\overline{Z} \cap A(\mathcal{P})| + |K| + k(|V| - |P|). \tag{28}$$

Let $\mathcal{P}_h := \{X \in \mathcal{P} : r_v(\partial^Z(v)) \leq k - h(v)\ (\forall v \in V)\}$, where $h \in \{f, g_k\}$. Then $\mathcal{P}_h$ is a subpartition of V and for every $X \in \mathcal{P} - \mathcal{P}_h$ there exists a $v_X \in X$ such that $r_v(\partial^Z(v)) > k - h(v)$.

By the definition of K, we have

$$A_{E(\mathcal{P}_h) - K} \subseteq Z \cap A_{E(\mathcal{P}_h)}. \tag{29}$$

Then, by (28), the definition of $\mathcal{P}_h$ and v_X, $r_v(\partial^Z(v)) \geq 0$, $k - h \geq 0$, $h \geq 0$ and $r(X) \leq |X|$ (subcardinality of the rank function of a matroid), we have:

$$r_{M_F^k}(\overline{Z}) + \sum_{v \in V} \min\{r_v(\partial^Z(v)), k - h(V)\} =$$

$$= |\overline{Z} \cap A(\mathcal{P})| + |K| + k(|V| - |\mathcal{P}|) + \sum_{v \in \cup \mathcal{P}} \min\{r_v(\partial^Z(v)), k - h(V)\} + \sum_{v \in \overline{\cup \mathcal{P}}} \min\{r_v(\partial^Z(v)), k - h(V)\}$$

$$\geq |\overline{Z} \cap A(\mathcal{P}_h)| + \sum_{v \in \cup \mathcal{P}} r_v(\partial^Z(v)) + \sum_{X \in \mathcal{P} - \mathcal{P}_h} \sum_{v \in X} \min\{r_v(\partial^Z(v)), k - h(V)\} + |K| + k(|V| - |\mathcal{P}|)$$

$$\geq |\overline{Z} \cap A(\mathcal{P}_h)| + r((Z \cap A(\mathcal{P}_h)) \cup (Z \cap A_{E(\mathcal{P}_h)})) + \sum_{X \in \mathcal{P} - \mathcal{P}_h} (k - h(v_X)) + |K| + k(|V| - |\mathcal{P}|)$$

$$\geq r(\overline{Z} \cap A(\mathcal{P}_h)) + r((Z \cap A(\mathcal{P}_h)) \cup (Z \cap A_{E(\mathcal{P}_h)})) + \sum_{X \in \mathcal{P} - \mathcal{P}_h} (k - h(X)) + |K| + k(|V| - |\mathcal{P}|)$$

$$\geq r(\overline{Z} \cap A(\mathcal{P}_h)) + r((Z \cap A(\mathcal{P}_h)) \cup (Z \cap A_{E(\mathcal{P}_h)})) + k(|\mathcal{P}| - |\mathcal{P}_h|) - h(\overline{\cup \mathcal{P}_h}) + |K| + k(|V| - |\mathcal{P}|)$$

$$= r(\overline{Z} \cap A(\mathcal{P}_h)) + r((Z \cap A(\mathcal{P}_h)) \cup (Z \cap A_{E(\mathcal{P}_h)})) - k|\mathcal{P}_h| - h(\overline{\cup\mathcal{P}_h}) + |K| + k|V|$$

Using (29) and the submodularity of r, we get

$$r(\overline{Z}\cap A(\mathcal{P}_h))+r((Z\cap A(\mathcal{P}_h))\cup(Z\cap A_{E(\mathcal{P}_h)})) \geq r(A(\mathcal{P}_h)\cup(Z\cap A_{E(\mathcal{P}_h)}))+r((\overline{Z}\cap A(\mathcal{P}_h))\cap(Z\cap A_{E(\mathcal{P}_h)}))).$$

$$\geq r(A(\mathcal{P}_h) \cup A_{E(\mathcal{P}_h)-K}) + r(\emptyset) \geq R(\mathcal{P}_h) - |K|.$$

In the last inequality we use $r(X - K) \geq r(X) - |K|$. Using the previous two inequalities, we get

$$r_{M_F^k}(\overline{Z}) + \sum_{v\in V} \min\{r_v(\partial^Z(v)), k - h(V)\} \geq R(\mathcal{P}_h) - k|\mathcal{P}_h| - h(\overline{\cup\mathcal{P}_h}) + k|V|.$$

Using this inequality for $h = f$ and (24) we get (26), and if we apply it for $h = g_k$, we get (27). □

Finally, we prove Theorem 5.4.

Proof. Necessity. The neccesity of $g_k \geq f$ and $\min\{\widetilde{g}_k(V), l'\} \geq l$ is trivial. Let $\mathcal{P}$ be a subpartition of V and let B be the arc set of an (f, g)-bounded k-regular (l, l')-limited M-restricted packing of arborescences in an orientation $\overrightarrow{F}$ of F. For a node v, let the number of v-arborescences in the packing be $q(v)$. Let C be a class of $\mathcal{P}$. By k-regularity, there is at least k arborescences in the packing, which have arcs induced by C. If the root of an arborescence is not in C, then it enters it. Thus, the number of edges in B that enter C is at least $k - \sum_{v\in C} q(v)$. The number of edges in B entering a class of $\mathcal{P}$ is therefore at least $k|\mathcal{P}| - \sum_{C\in\mathcal{P}} \widetilde{q}(C) = k|\mathcal{P}| - \widetilde{q}(\cup\mathcal{P})$. Since the packing is (f, g)-bounded and (l, l')-limited, we have $\widetilde{q}(\cup\mathcal{P}) \leq \min\{l' - f(\overline{\cup\mathcal{P}}), \widetilde{g}_k(\cup\mathcal{P})\}$, therefore the right side of (24) is a lower bound on the number of edges in B, that enter a member of $\mathcal{P}$. Since B is independent in M, we get (24).

Sufficiency. Let $(F = (V, E \cup A), f, g, k, l, l')$ an instance of Theorem 5.4, that satisfies the necessary conditions. Since (24) holds, by Theorem 5.5/(C), (25) and (26) hold. Since $g_k \geq f$ and $\min\{\widetilde{g}_k(V), l'\} \geq l$ also hold, by Theorem 5.5/(B), T (as defined in Theorem 5.5) is nonempty, thus, by Theorem 5.1/2./(ii), it contains an integral element x. By Theorem 5.5/(A), x is the characteristic vector of the edge sets of an (f, g)-bounded k-regular (l, l')-limited M-restricted packing of arborescences in an orientation $\overrightarrow{F} = (V, \overrightarrow{E} \cup A)$ of F. Replacing the arcs in $\overrightarrow{E}$ with the edges in E, we get the required packing. □

If we choose the matroid in Theorem 5.4 to be a partition matroid, we can prescribe bounds on the in-going directed edges in the packing:

Collorary 5.1. *Let $F = (V, E \cup A)$ be a mixed graph, $f, g, h : V \to \mathbb{Z}_+$ functions and $k, l, l' \in \mathbb{Z}_+ - \{0\}$. There exists an (f, g)-bounded k-regular (l, l')-limited packing of arborescences in F with $\varrho_{A\cap T}(v) \leq h(v)$ for every $v \in V$ where T is the edge set of the packing, if and only if $g_k(v) \geq f(v)$ for every $v \in V$, $\min\{\widetilde{g}_k(V), l'\} \geq l$ and for every subpartition $\mathcal{P}$ of V*

$$\sum_{v\in\bigcup_{i=1}^q V_i} \min\{m_{in}(v), |\partial(v) \cap \partial(V_i)|\} \geq k|\mathcal{P}| - \varrho_E(\mathcal{P}) - \min\{l' - f(\overline{\cup\mathcal{P}}), \widetilde{g}_k(\cup\mathcal{P})\} \qquad (30)$$

Proof. For every $v \in V$, let M_v a partition matroid with partition classes $\partial_A(v)$ and $\partial_E(v)$, and bounds $h(v)$ and $\varrho_E(v)$ (that is, $M_v|\partial_E(v)$ is the free-matroid) and let $M := \bigoplus_{v\in V} M_v$. Then, an M-restricted packing with edge set T satisfies $\varrho_{A\cap T}(v) \leq h(v)$.

Let R be as defined in Theorem 5.4. For a subpartition $\mathcal{P} = \{V_1, \ldots, V_q\}$ of V:

$$R(\mathcal{P}) = \varrho_E(\mathcal{P}) + \sum_{v\in\bigcup_{i=1}^q V_i} \min\{m_{in}(v), |\partial(v) \cap \partial(V_i)|\} \qquad (31)$$

Therefore (4.1) is equivalent to (24) with f, g, k, l, l' and the matroid M which proves the statement.

□

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