Free-rooted arborescence packings

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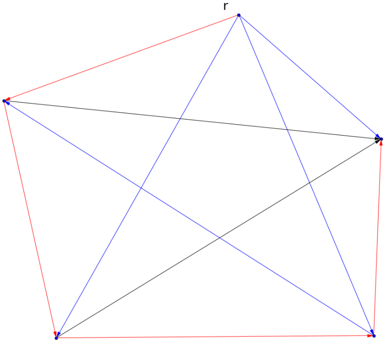
Arborescence packings

– arborescence: directed tree in which there is a directed path to every vertex from a root
– branching: every connected component is an arborescence
–arborescence/branching-packing: a set of edge-disjoint arborescences/branchings



Theorem (Edmonds, weak form)

Let $D = (V, A)$ be a digraph, $r \in V$ a vertex and k a positive integer. When does D contain k edgedisjoint r-rooted spanning arborescences?

Arborescence packings

Theorem (Edmonds, strong form)

Let $D = (V, A)$ be a digraph and let $R_1, \dots, R_k$ be subsets of V. When does D contain k edge-disjoint spanning branchings, with root-sets $R_1, \dots, R_k$?

size of the root-set $\leftrightarrow$ number of edges in the branching

free-rooted packings:

Theorem (Bérczi, Frank)

Let $D = (V, A)$ be a digraph let $\mu_1, \dots, \mu_k$ positive integers. When does D contain k edge-disjoint spanning branchings with the given sizes?

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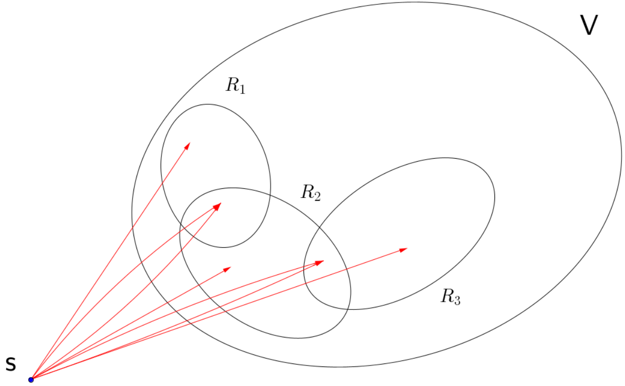
Equivalent formulation of the problem

Let s be a vertex not in V with an in-degree 0. Partition the outgoing edges (root-edges), the endpoints in the partition classes correspond to the R_i sets.



Matroids

a structure that generalizes linear independence

multiple definitions (independent subsets, bases, rank)

k-uniform matroid, partition matroid

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Matroid based arborescence packings

Let M_1 be a matroid on the root-edges, and M_2 a matroid on A.

Definition

An M_1-**based** s-**arborescence packing** is a collection of $T_1, \dots T_k$ edge-disjoint s-arborescences, for which every T_i contains exactly one root-edge e_i, and for all $v \in V$ vertices the set $\{e_i : v \in T_i\}$ forms a base of the matroid M_1.

Definition

M_2-**restricted** s-**arborescence packing** is a collection of $T_1, \dots T_k$ edge-disjoint s-arborescences, for which the set $\bigcup_{i=1}^{q} T_i$ is independent in M_2.

Matroid based arborescence packings

Theorems used:

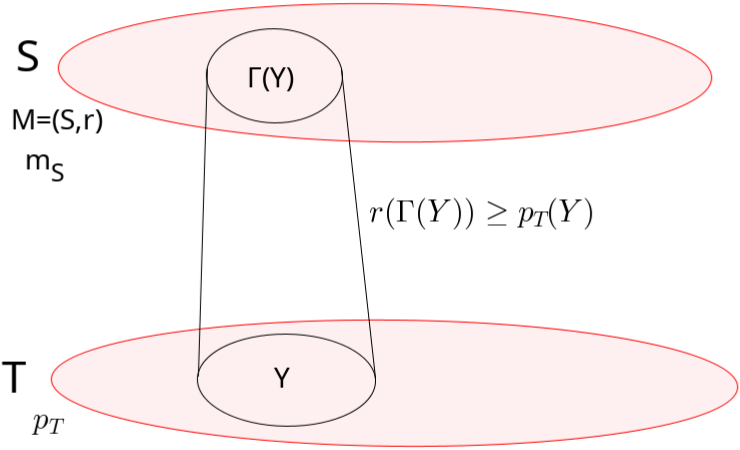
Theorem (Bérczi, Frank)

Let $M = (S, r)$ be a matroid, p_T a positively intersecting supermodular setfunction on T and m_S an degree-prescription on S. When can we construct a simple bipartite graph $G = (S, T, E)$, which satisfies the degree prescription and M-covers p_T, that is

$$r(\Gamma(Y)) \geq p_T(Y) \; \forall Y \subseteq T$$

?

Matroid based arborescence packings



Matroid based arborescence packings

Theorem (Cs. Király, Szigeti, Tanigawa)

Let $D = (V + s, A)$ be a digraph and M_1 a matroid on the root-edges. Moreover, let M_v be a matroid on $\partial(v)$ for all $v \in V$ and let M_2 be the direct sum of the M_v matroids. When does D contain an M_1-based M_2-restricted s-arborescence packing?

Free rooted packings:

Theorem

We can add edges from s to V and map the elements of S to these edges such that there exists an M_1-based M_2-restricted s-arborescence packing if and only if

$$(k - r_1(X))q - |S - X| \leq \sum_{i=1}^{q} r_2(\partial(V_i))$$

for all $\{V_1, \ldots, V_q\}$ subpartitions of V and $X \subseteq S$-re.

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Matroid based arborescence packings

Let $M = (S, r)$ be a matroid with a rank function r and let $D = (V, A)$ be a digraph with an $m_{in} : V \to \mathbb{Z}^+$ indegree-prescription for which $0 \leq m_{in}(v) \leq \varrho_D(v)$ and $m_{in}(V) \leq r(M)$ holds for all $v \in V$. Moreover, $\widetilde{m}_{in}(V) = |V|r(M) - |S|$. Let s be a vertex not in V.

Theorem

The following statements are equivalent:

(A) *We can add edges from s to V and map the elements of S to these edges such that there exists an M-based s-arborescence packing which satisfies the indegree-prescription m_{in}*

Matroid based arborescence packings

Theorem

(B1) *For all subpartition* $\{V_1, \ldots, V_q\}$ *of* V

$$(r(M)-r(X))q-|S-X| \leq \sum_{v \in \bigcup_{i=1}^{q} V_i} \min\{m_{in}(v), |\partial(v) \cap \partial(V_i)|\}$$

(B2) *For all* $Y \subseteq V$, *subpartition* $\{V_1, \ldots, V_q\}$ *of* $V - Y$ *and* $X \subseteq S$*-re:*

$$(|Y| + q)(r(M) - r(X)) - |S - X| \leq \widetilde{m}_{in}(Y) + \sum_{i=1}^{q} \varrho_D(V_i)$$

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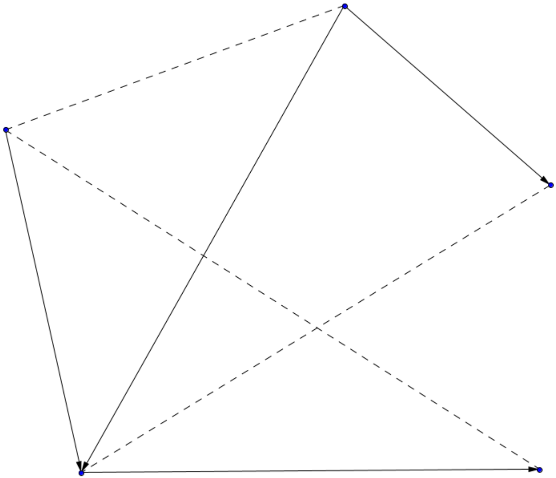
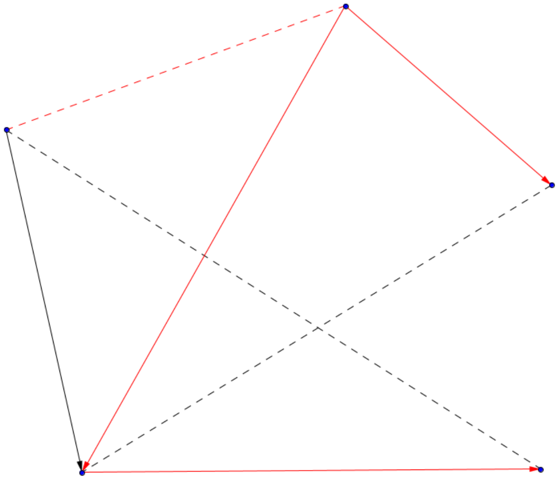
Arborescence packings in mixed graphs



Arborescence packings in mixed graphs



Arborescence packings in mixed graphs

Let $F = (V, E \cup A)$ be a mixed graph. Let A_E be the set of directed edges we get by directing the edges of E both ways. Let $f, g : V \to \mathbb{Z}_+$ be functions and k, l, l' positive integers.

Theorem (Szigeti)

The following statements are equivalent:

(A) *F contains an (f, g)-restricted, k-regular, (l, l')-limited arborescence packing.*

(B) *(...) and for every subpartition $\mathcal{P}$ of V*

$$\varrho_{A \cup E}(\mathcal{P}) \geq k|\mathcal{P}| - \min\{l' - f(\overline{\cup \mathcal{P}}), \widetilde{g}_k(\cup \mathcal{P})\}.$$

Arborescence packings in mixed graphs

Let $F = (V, E \cup A)$ be a mixed graph. Let A_E be the set of directed edges we get by directing the edges of E both ways. Let $f, g : V \to \mathbb{Z}_+$ be functions and k, l, l' positive integers. Moreover, let M_v be a matroid on $\partial_{A \cup A_E}(v)$ and let M be the direct sum of these matroids.

Theorem

Ekvivalensek:

(A) *F contains an (f, g)-restricted, k-regular, (l, l')-limited M-restricted arborescence packing.*

(B) *(...) and for every subpartition $\mathcal{P}$ of V*

$$R(\mathcal{P}) \geq k|\mathcal{P}| - \min\{l' - f(\overline{\cup\mathcal{P}}), \widetilde{g}_k(\cup\mathcal{P})\}.$$

where $R(\mathcal{P}) = \max\{r(\overrightarrow{\partial_{A \cup E}(\mathcal{P})})\}$, where $\overrightarrow{\partial_{A \cup E}(\mathcal{P})}$ is an orientation of $\partial_{A \cup E}(\mathcal{P})$-nek.

Arborescence packings in mixed graphs

Collorary:

$M_v|\partial_A(v)$ is an uniform matroid for all $v \in V \longrightarrow$ upper bound on the in-going directed edges in the packing for each edge

Thank you for your attention!