# Quantum Acceleration for the American Option Problem

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# The American Option Problem

- Option: Right to buy/sell a product at a predetermined price at some future date.

- European Style Option: can only be exercised at the expiry date.
- American Style Option: can also be exercised before the expiry date.

- Problem: Suppose we model the possible market changes with probability distributions.
  Given the stochastic process with different profits at time $t$, when to exercise to have max $\mathbb{E}(\text{profit})$?

# Problem Modeling

- $t = 0, ..., T$ discrete times
- $(X_t)_{t=0}^{T}$ Markov-chain with $\Omega$ sample space and $E \subseteq \mathbb{R}^d$ state space.
- $(Z_t)_{t=0}^{T}$ payoff process $Z_t := z_t(X_t)$ where $z_t \in L_2(E)$
- $U_t$: expected payoff if we get to time $t$
- $\tau_t$: stopping time if we get to time $t$
- Goal: approximate $U_0$ and $\tau_0$

# Problem Modeling

Snell envelope:

$$U_t := \begin{cases} Z_T & \text{if } t = T \\ \max\{Z_t, \mathbb{E}(U_{t+1}|X_t)\} & \text{else} \end{cases}$$

$$\tau_t := \begin{cases} T & \text{if } t = T \\ t \cdot \mathbb{1}\{Z_t \geq \mathbb{E}(Z_{\tau_{t+1}}|X_t)\} + \tau_{t+1} \cdot \mathbb{1}\{Z_t < \mathbb{E}(Z_{\tau_{t+1}}|X_t)\} & \text{else} \end{cases}$$

# Least Squares Monte Carlo Method

- Take $N$ independent samples: $(X_t^{(1)})_{t=0}^T, \dots, (X_t^{(N)})_{t=0}^T$
- Approximation scheme: $\{e_{t,k}\}_{k=1}^m$ lin. indep. functions from $L_2(E)$
- We want to estimate $\mathbb{E}(U_{t+1}|X_t)$ in this scheme:

$$\alpha_t = \arg\min_a \mathbb{E}\big((U_{t+1} - a \cdot e_t(X_t))^2\big)$$

**Remark**

Let $A$ be the $m \times m$ covariance matrix where $\big(A_t\big)_{i,j} = \mathbb{E}\big(e_{t,i}(X_t)e_{t,j}(X_t)\big)$ and $b_t = \mathbb{E}\big(U_{t+1}e_t(X_t)\big)$. Then $\alpha_t \approx A_t^{-1} \cdot b_t$.

# Classical Algorithm

## LSM

1. Take sample simulations: $(X_t^{(i)})_{t=0}^{T} \quad i = 1, ..., N$
2. Calculate values $Z_t^{(i)}$ and $e_{t,k}(X_t^{(i)})$ for all
   $t = 0, ..., T \; ; \; i = 1, ..., N \; ; \; k = 1, ..., m$
3. Specify $\widetilde{A}_t$ for each $t = 0, ..., T$ and calculate their inverses as well.
4. $\tilde{u}_T = z_T \qquad \forall \; i = 1, ..., N$

   **FOR** $t = (T-1), ..., 0$:

   $$\tilde{\alpha}_t = \widetilde{A}_t^{-1} \frac{1}{N} \sum_{i=1}^{N} \tilde{u}_{t+1}(X_{t+1}^{(i)}) \cdot e_t(X_t^{(i)})$$
   $$\tilde{u}_t := \max\{z_t, \tilde{\alpha}_t \cdot e_t\}$$
5. **RETURN**

$$\widetilde{U}_0 = \frac{1}{N} \sum_{i=1}^{N} \tilde{u}_0(X_0^{(i)})$$

# Expected Value Estimation

$y$ discrete real random variable, $\mu = \mathbb{E}(y)$

Goal: approx $\mu$ with $\hat{\mu}$ for given $\delta$, $\epsilon$ where

$$\mathbb{P}\big(|\hat{\mu} - \mu| > \epsilon\big) \leq \delta$$

How many samples do we need?

## Classical approach

Sampling $y$ $N$ times, then output their mean. By Chebyshev's ineq.:

$$\mathbb{P}\big(|\hat{\mu} - \mu| > \epsilon\big) \leq \frac{\sigma^2}{N\epsilon^2}$$

for constant $\delta$: $N = O(\sigma^2/\epsilon^2)$

In quantum setup $O(\sigma/\epsilon)$ sample can achieve the same.

# Quantum LSM

**P** is a quantum gate: $\mathbf{P}|0\rangle = \sum \sqrt{p(x)}|x\rangle$
where
$p(x) = \mathbb{P}(X_1 = x_1)\mathbb{P}(X_2 = x_2|X_1 = x_1)...\mathbb{P}(X_T = x_T|X_{T-1} = x_{T-1})$

### Kothari-O'Donnell (2022)

With **P** expected values can be estimated by $\hat{N} := O(\sigma/\epsilon)$ queries.

# Quantum LSM Algorithm

## QLSM

1. Estimate $e_{t,k}(X_t)e_{t,l}(X_t)$ values by $\hat{N}$ queries (i.e. with $\hat{N}$ number of **P** calls)
2. By this specify all $\widetilde{A}_t$, and their inverses too classically
3. $\tilde{u}_T := z_T$
4. **FOR** $t = (T-1), ..., 0$:
   Estimate $\tilde{u}_{t+1} \cdot e_{t,k}(X_t)$ then specify $\tilde{b}_t$ (here also with $\hat{N}$ calls)
   $\tilde{\alpha}_t = \widetilde{A}_t^{-1} \cdot \tilde{b}_t$
   $\tilde{u}_t := \max\{z_t, \tilde{\alpha}_t \cdot e_t\}$
5. **RETURN** $\tilde{u}_0$

# Runtime Comparision

- Classic LSM:

$$O(N \cdot \mathbf{T}_{samp} + N \cdot T \cdot m^2 + T \cdot m^\omega)$$

- Quantum LSM:

$$O(\hat{N} \cdot \mathbf{T}_{qsamp} \cdot T \cdot m^2 + T \cdot m^\omega)$$

# Kothari-O'Donnell

$y$ discrete real random variable, $\mu = \mathbb{E}(y)$. Kell: $\hat{\mu} : \mathbb{P}\big(|\hat{\mu} - \mu| > \epsilon\big) \leq \delta$

**Main Theorem:**

There is a quantum algorithm that using $O(n)$ samples outputs $\hat{\mu}$ where:

$$\mathbb{P}\left(|\hat{\mu} - \mu| \geq \frac{\sigma}{n}\right) \leq \frac{1}{3}$$

By reduction it is enough to prove:

**Theorem 2.:**

Given $\epsilon > 0$ suppose that $\mathbb{E}(y^2) \leq 1$, then there exists a quantum algorithm, that uses $O(1/\epsilon)$ samples and with probability $2/3$ distinguishes *a*) $|\mu| \leq \epsilon/2$, and *b*) $\epsilon \leq |\mu| \leq 2\epsilon$.

# Kothari-O'Donnell

- $\mathbf{P}|0\rangle = \sum_{\ell=1}^{D} \sqrt{p(\ell)} \cdot |\ell\rangle |garbage_\ell\rangle$
- $\alpha_\ell := -2\ arctan(y_\ell)$
- $\mathbf{U} := REFL_p \cdot ROT_y$, where

  $REFL_p := \mathbf{P}(2|0\rangle\langle 0| - I)\mathbf{P}^\dagger$
  and
  $ROT_y$ such as $\quad ROT_y|\ell\rangle|garbage_\ell\rangle = e^{i\alpha_\ell}|\ell\rangle|garbage_\ell\rangle$

# Kothari-O'Donnell

## Notation

- $\sum_{j=1}^{D} e^{i\theta_j}|u_j\rangle\langle u_j|$ is an eigendecomposition of $\mathbf{U}$ and $|\sigma\rangle$ is a unitvector
- Then in this basis $|\sigma\rangle = \sum_{j=1}^{D} \hat{\sigma}_j |u_j\rangle$, where $\hat{\sigma}_j = \langle u_j|\sigma\rangle$
- $|\hat{\sigma}_1|^2, |\hat{\sigma}_2|^2, \ldots, |\hat{\sigma}_D|^2$ determines a probability distribution

Now $\theta \sim \Theta_{\mathbf{U}}(|\sigma\rangle)$ will denote that $j \in [D]$ index is chosen according to the distribution induced by $|\sigma\rangle$ and we choose $\theta_j$ from the eigendecomposition of $\mathbf{U}$.

# Kothari-O'Donnell

**Lemma 3.:**

Suppose that $\mathbb{E}(y^2) \leq 1/16$. If $\theta \sim \Theta_{\mathbf{U}}(\mathbf{P}|0\rangle)$, then:

$$\mathbb{P}(4/5 \cdot 2|\mu| \leq |\theta| \leq 5/4 \cdot 2|\mu|) \geq 1 - 2/9$$

For **Theorem 2.** recall we want an $O(1/\epsilon)$ runtime quantum algorithm, which distinguishes with at most $1/3$ error between $|\mu| \leq \epsilon/2$, and $|\mu| > \epsilon/2$.

**Algorithm for Theorem 2.:**

1. Phase estimation on $\mathbf{P}|0\rangle$ with controlled-$\mathbf{U}$ gates, output: $\theta'$ (accuracy: $\epsilon/6$, error $\leq 1/9$)
2. **IF** $|\theta'| > 142/100 \cdot \epsilon$, then **RETURN**($|\mu| \leq \epsilon/2$)
   **ELSE RETURN**($|\mu| \geq \epsilon$)

# Kothari-O'Donnel

From lemma 3.:

$$\mathbb{P}(4/5 \cdot 2|\mu| \leq |\theta| \leq 5/4 \cdot 2|\mu|) \geq 1 - 2/9$$

With at most $1/9 + 2/9 = 1/3$ error:

$$4/5 \cdot |\mu| - \epsilon/12 \leq |\theta'/2| \leq 5/4 \cdot |\mu| + \epsilon/12$$

If $|\mu| \leq \epsilon/2$, then

$$|\theta'/2| \leq 5/8 \cdot \epsilon + \epsilon/12 < 71/100 \cdot \epsilon$$

And if $|\mu| \geq \epsilon$, then

$$|\theta'/2| \geq 4/5 \cdot \epsilon - \epsilon/12 > 71/100 \cdot \epsilon$$

Thank you for your attention!