Arc-partitioning and vertex-ordering problems

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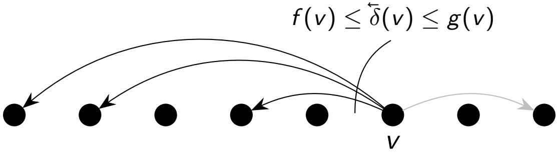
First two semesters: The $(f, g, \sum w)$-FAS problem

$(f, g; \sum w)$-FAS problem

Let us given a digraph $D = (V, A)$ with a weight function $w : A \to \mathbb{R}_+$, and lower and upper bound functions $f : V \to \mathbb{R}_+$ and $g : V \to \mathbb{R}_+$. Does there exist an order of the vertices such that the left weighted out-degree of each vertex v is between $f(v)$ and $g(v)$?

In the unweighted case, the problem is called the (f, g)-FAS problem.



Only upper bounds

Theorem

The $(-\infty, g, \sum w)$-FAS problem, where only upper bounds are given on the vertices, is polynomial-time solvable.

Algorithm: In each step, fix a vertex v which does not violate the upper bound $g(v)$ on the last free place, and delete v from D.
If no such vertex exists at some point, then no solution exists.

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If no such vertex exists at some point, then no solution exists.

The case when only lower bounds are given can be solved similarly.

Complexity

The previously discussed solvable cases become NP-complete after small modifications.

Theorem

The (f, g)-FAS problem is NP-complete if only upper bounds are given for all vertices except for a single vertex v for which $f(v) = g(v)$.

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Theorem

The $(-\infty, g; \sum w)$-FAS problem is NP-complete if negative arc-weights are allowed.

Partitioning into an in-branching and an acyclic subgraph

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A digraph can be partitioned into an in-branching and an acyclic subgraph if and only if the $(-\infty, g)$-FAS problem with $g \equiv 1$ is solvable.

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Theorem

It is NP-hard to find a minimum-size in-branching whose complement is acyclic.

Rank aggregation

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For a candidate v, let $d(v)$ denote the number of the candidates whom v precedes by the majority of the judges, but not in the common ranking.

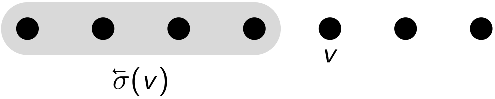
We can minimize the maximum distance $d(v)$ over all candidates by a reduction to the $(-\infty, g)$-FAS ordering problem.

Generalization

$(f, g; h)$-ordering problem

For a finite ground set V, let us given a non-decreasing set-function $h_v : 2^{V-v} \to \mathbb{R}$ for each item $v \in V$, and lower and upper bound functions $f : V \to \mathbb{R}$ and $g : V \to \mathbb{R}$.
Does there exists an order σ of V such that, for each item v, the set $\overleftarrow{\sigma}(v)$ of the items preceding v satisfies $f(v) \leq h_v(\overleftarrow{\sigma}(v)) \leq g(v)$?

The cases where only upper or only lower bounds are given are polynomial-time solvable by modifying the algorithms given for the $(f, g; \sum w)$-FAS problem.



Sceduling: $1|g\text{-prec}|L_{max}$

Let us given some jobs with

- $p(v)$: processing time of v
- $d(v)$: due date of v
- „precedence" constraints (not necessarily acyclic)
- $g(v)$: upper bound for the number of the violated precedence constraints for v.

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Consider the $(-\infty, g; h)$-ordering problem with
$h_v(V') = M(\delta(v, V') - g(v))^+ + (\sum_{u \in V'} p(u) + p(v) - d(v))^+$
$g' \equiv K$.
We can decide in polynomial time, whether there exists a schedule with maximum lateness K.

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- $\min_{v \in V} \sum_{v \in V} h(\overleftarrow{d}(v))$ problem: For a given strictly convex function $h : \mathbb{Z}_+ \to \mathbb{R}$, the goal is to minimize the $\sum_{v \in V} h(\overleftarrow{d}(v))$ function.

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Each vertex-ordering problem above can be equivalently rephrased as an acyclic orientation problem.

Egalitarian *orientation* problems

Let us given an undirected graph G. Our goal is to find an orientation D of G such that the indegree vector of D is as "equitable" as possible.

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- $\min_{v \in V} \sum_{v \in V} h(\varrho(v))$ problem: For a given strictly convex function $h : \mathbb{Z}_+ \to \mathbb{R}$, the goal is to minimize the $\sum_{v \in V} h(\varrho(v))$ function.

Theorem (Frank, Murota (2022))

The optimal orientations coincide for the dec-min, the inc-max and the $\min \sum_{v \in V} h(\varrho(v))$ *orientation problems.*

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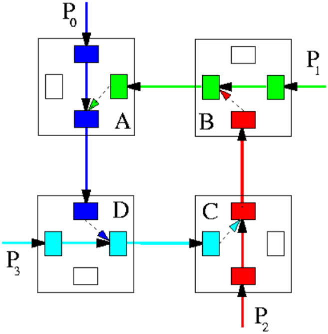
The $\min \sum_{v \in V} h_v(\varrho(v))$ orientation problem is also solvable.

Application: Routing protocol

Let us given a G=(V,E) undirected network, with capacities on the vertices, and packets with start and end destinations.

Such networks can suffer form deadlock if the packets are waiting in a cycle in which every vertex reached its capacity.

Routing protocol: Consider an order of the vertices, and forbid the packets to use the length-two paths formed by edges that are going to the left from the same vertex.



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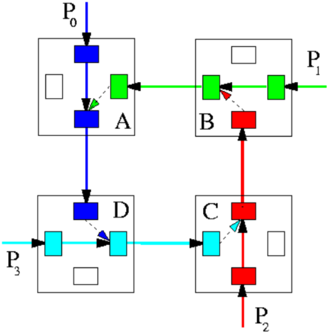

- dec-min: Minimizes the maximum number of forbidden length-two paths for a vertex.
- $\min \sum_{v \in V} \overleftarrow{d}^2(v)$: Minimizes the number of forbidden length-two paths.

The complexities of the dec-min and inc-max problems

We call an order k-bounded, if every left degree is at most k (i.e. it is an $(-\infty, g)$-FAS order for $g \equiv k$).

Theorem (Borradaile, Iglesias, Migler, Ochoa, Wilfong, Zhang (2017))

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The dec-min and the inc-max ordering problems are NP-hard, even in case of k-bounded orders for all $k \geq 3$.

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The dec-min and the inc-max ordering problems are NP-hard, even in case of k-bounded orders for all $k \geq 3$.

The dec-min and the inc-max ordering problems coincide in case of 2-bounded orders, but the complexity remains open.

The min $\sum_{v \in V} h(\overleftarrow{d}(v))$ ordering problem

Let us given a strictly convex function $h : \mathbb{Z}_+ \to \mathbb{R}$ and an undirected graph $G = (V, E)$.
Goal: Finding an order that minimizes the objective function $\sum_{v \in V} h(\overleftarrow{d}(v))$.

Important strictly convex functions:

- $h(z) = |V|^z$: dec-min ordering problem,
- $h(z) = |V|^{-z}$: inc-max ordering problem,
- $h(z) = z^2$.

The complexity of the min $\sum_{v \in V} h(\overleftarrow{d}(v))$ ordering problem

Theorem

For any strictly convex function $h : \mathbb{Z}_+ \to \mathbb{R}$, it is NP-hard to find a vertex order minimizing $\sum_{v \in V} h(\overleftarrow{d}(v))$ in case of multigraphs.

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Theorem

For any linear functions $h_v : \mathbb{Z}_+ \to \mathbb{R}$ the problem of finding a vertex order minimizing $\sum_{v\in V} h_v(\overleftarrow{d}(v))$ is polynomial-time solvable.

Exact dynamic program for $\min \sum_{v \in V} h_v(\overleftarrow{d}(v))$

Let $f(V')$ denote the optimum for the graph induced by the vertex set V'.

$$f(V') = \min_{v \in V'} \{ f(V' - v) + h_v(d(v, V')) \}$$

Theorem

We can compute the value of $f(V')$ for every subset $\emptyset \neq V' \subseteq V$ in non-decreasing order by $|V'|$ in $O(2^{|V|} poly(|V|, |E|))$ time, provided that the function h_v can be evaluated in polynomial time for each $v \in V$.

The min $\sum_{v \in V} \overleftarrow{d}^2(v)$ ordering problem

Greedy algorithm: Fix a vertex with minimum degree at the last free place in the order and delete it from G until no vertex remains.

Theorem

The greedy order gives a $\min\{4H_n, k_{\min}\}$*-approximate order, where* H_n *is the* n^{th} *harmonic number and* $k_{\min}$ *denotes the smallest integer* k *for which the graph has a* k*-bounded order.*

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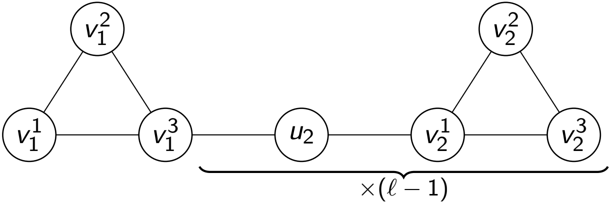

A simple graph, for which the greedy order is $\frac{9}{7}$-approximate if $\ell \to \infty$.

“Balanced” vertex orders

The goal is to find a vertex order for which $\overleftarrow{d}(v) \approx \overrightarrow{d}(v)$.
A vertex order is perfectly balanced if $|\overleftarrow{d}(v) - \overrightarrow{d}(v)| \leq 1 \ \forall v \in V$.

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Theorem (Biedl, Chan, Ganjali, Hajiaghayi, Wood (2005) and Kára, Kratochvíl, Wood (2005))

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For graphs with maximum degree 3, we can find a perfectly balanced vertex order in polynomial time.

The max $\sum_{v \in V} \overleftarrow{d}(v)\overrightarrow{d}(v)$ problem

Goal: Find a vertex order maximizing $\sum_{v \in V} \overleftarrow{d}(v)\overrightarrow{d}(v)$.

Corollary

A max $\sum_{v \in V} \overleftarrow{d}(v)\overrightarrow{d}(v)$ *problem is NP-hard, even for graphs with maximum degree 4.*

Theorem

For graphs with maximum degree 3, we can find a vertex order maximizing $\sum_{v \in V} \overleftarrow{d}(v)\overrightarrow{d}(v)$ *in polynomial time.*

Approximation

Theorem

A random vertex order is a 3-approximate solution in expectation.

This estimate is sharp for a star with 3 leaves.

We gave a de-randomized 3-approximating algorithm.

Open questions

- Partitioning a digraph into an in-arborescence and an acyclic subgraph.
- The complexity of the 2-bounded dec-min and inc-max problems.
- The complexity of the min $\sum_{v \in V} h(\overleftarrow{d}(v))$ problem
  - for simple graphs, and
  - for special graph classes (e.g. planar graphs).
- The approximation ratio of the greedy order for minimizing the square-sum of the left degrees.