Eötvös Loránd University

Faculty of Sciences

Institute of Mathematics

Well-posedness of the Heat Equation on Non-Compact graph with non-local-Kirchhoff-type condition

Supervisor: Eszter Sikolya

Author: CHTIBA REDA
Msc in APPLIED MATHEMATICS

Budapest, 12/19/2022

Introduction

The field of Evolutionary Equations on Metric Graphs has been studied since the first decades of the 20th century and receiving more attention recently thanks to their usefulness in many new interesting problems in Theoretical Physics, biology, and engineering. For example, such graphs arise naturally when considering the propagation of the heat through a quasi-one-dimensional system that locally looks like a graph. This field of study is also called Dynamics on Networks or one-dimensional ramified spaces or known in some communities of theoretical physics as quantum graphs. Working in this field and its applications brings together many nice tools and ideas from various branches such as graph theory, PDEs, mathematical physics, and many other areas depending on the type of problem.
Some of the studies that have been worked on in this area are related to the structure of the graph and the type of boundary conditions of PDEs which have physical interpretations. For example, in [7] and [8], there are results concerning the well-posedness of some types of 2nd-order PDEs with non-local Kirchhoff-type conditions on compact graphs, where in both papers they used similar functional analysis tools to study the well-posedness of such PDEs on Networks. While in [3], they treated 2nd order PDEs in a much more general setting, which includes very general types of boundary conditions and they even considered non-compact graphs as well. The contrast between the first two papers and the third one is that the methods they used to prove the well-posedness are different. Therefore the aim of this project is to use a similar method as [7] and [8]to prove the well-posedness of the Heat Equation on a non-compact graph with the same type of boundary conditions that were treated in this paper [5], and also to equivalently prove the well-posedness by showing that these boundary conditions can be reformulated to fit in the framework considered in [3] and use directly its well-posedness result
In this report, we will introduce in the first chapter, the main notions, and necessary background from graph theory, semigroup theory, and some functional analysis. In the second chapter, we will introduce the framework that we will work with and prove the well-posedness using the two different methods mentioned earlier.

Contents

1 Background and Terminology **4**

1.1 Some Graph Theory 4

1.2 Semigroup Operator Theory and some Functional Analysis 5

1.2.1 Some functional analysis reminder: 5

1.2.2 Background on Semigroup theory: 5

1.2.3 Semigroups, sesquilinear forms, and operators: 5

2 Heat Equation on non-compact Graphs **7**

2.1 Introduction and Framework: 7

2.2 Boundary Spaces, Operators, and the ACP 8

2.3 Well-posedness of the abstract Cauchy problem: 9

2.3.1 First Method: 9

2.3.2 Second Method: 14

1

Background and Terminology

1.1 Some Graph Theory

Types of Graphs:

A Graph **G** consisting of a finite or countably infinite set of vertices **V**=$\{v_l\}$, a set **E**=$\{e_l\}$ of edges and a mapping ϕ from **E to VxV**, called the incidence mapping associated with G is said to be:

Simple: if it contains no loops and no multiple edges.

Directed: if $\phi(e) = (v, w)$ is an ordered pair where $e \in$ **E** and $v, w \in$ **V**; we say e is a directed edge from its tail v to its head w.

Metric: If each directed edge e is assigned a positive length $L_e \in [0, \infty)$ and parameterized on $[0, L_e]$ i.e., $x_e(s) \in e$ for $s \in [0, L_e]$ and if $\phi(e) = (v, w)$ then $x_e(0) = v$ and $x_e(L_e) = w$. The lengths of each directed edge that are reversal to each other are assumed to be equal. For the sake of notation simplicity, we denote $x_e(s)$, $x_e(0)$ and $x_e(L_e)$ by $e(s)$, $e(0)$ and $e(L_e)$ respectively.

A Metric Graph G is said to be **Equilateral:** if all the edges are assigned the same positive length. **Normalized:** if G is Equilateral and $L_e = 1$ for all edges. **Finite:** if the number of edges and vertices is finite. **Non-Compact:** if some edges have infinite length i.e parameterized on $[0, \infty)$, they are called leads and they are incident to only one vertex.

We consider such metric graphs because their structure in taking the union of all edges enables us to think about it as a topological space, in this way we would have the original vertices that are useful in identifying certain relations between the edges but also additional intermediate points on the edges which are helpful because, this way, we can consider functions on each edge or by taking a vector-valued function we can define such function on the whole graph more specifically on the topological space induced from it. (More details about Metric graphs can be found [10],[1]).

Incidence Matrices:

Let G=(V,E,ϕ) be a Finite-Equilateral-Normalized-Metric Graph (Abbreviation:(FENM)) such that V=$\{v_i\}_{i=\overline{1,n}}$ and E=$\{e_j\}_{j=\overline{1,m}}$ where $n, m \in \mathbb{N}$, as mentioned before ϕ describes the incidence relations between the vertices and edges, such mapping induces two matrices Φ^+=$(\phi^+_{ij})_{n\times m}$ and Φ^-=$(\phi^-_{ij})_{n\times m}$, each describes in the incidence relation in the following way:

$$\phi_{ij}^{+} := \begin{cases} 1, & \text{if } \mathrm{e}_j(0) = \mathrm{v}_i, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \phi_{ij}^{-} := \begin{cases} 1, & \text{if } \mathrm{e}_j(1) = \mathrm{v}_i, \\ 0, & \text{otherwise} \end{cases}$$

Here Φ^+ and Φ^- are called the outgoing and incoming incidence matrix respectively, accordingly, we call e_j an outgoing edge for v_i if $\phi_{ij}^{+} = 1$ and incoming edge for v_i if $\phi_{ij}^{-} = 1$.

1.2 Semigroup Operator Theory and some Functional Analysis

1.2.1 Some functional analysis reminder:

Definition 1.2.1.1 : ([2] , Chapter 1,Definition 6.1) If $(H_1, \langle .,. \rangle_{H_1})$ and $(H_2, \langle .,. \rangle_{H_2})$ are two Hilbert spaces, then the product of $H_1 \times H_2$ is also a Hilbert space with

$$\langle .,. \rangle_{H_1 \times H_2} = \langle .,. \rangle_{H_1} + \langle .,. \rangle_{H_2}$$

Definition 1.2.1.2 : ([4] , Definition 3.13) A linear operator $(A, D(A))$ on a Banach space X is called dissipative if

$$\|(\lambda - A)x)\| \geq \lambda \|x\| \text{ for all } \lambda \geq 0, x \in D(A)$$

Proposition 1.2.1.3 : ([2] , Chapter 10,Proposition 1.6) If A is a densely defined operator then the adjoint of A is a closed operator.

1.2.2 Background on Semigroup theory:

Definitions 1.2.2.1 : ([4],Chapter 1 and 2)

$C_0 -$ *Semigroup:* Let $(T(t))_{t\geq 0}$ be a family of bounded linear operators on a Banach space X. We call $(T(t))_{t\geq 0}$ a strongly continuous semigroup if :

$$T(t+s) = T(t)T(s) \quad \forall t,s \geq 0; \quad T(0) = Id; \quad \forall x \in X \quad t \mapsto T(t)x \in X \quad \text{is continuous}$$

Generator: The generator $A : D(A) \subseteq X \to X$ of a $C_0 -$ *Semigroup* $(T(t))_{t\geq 0}$ on X is the operator defined by:

$$\begin{cases} Ax := \lim_{h\to 0} \frac{1}{h}(T(t)x - x) \\ D(A) := \{x \in X : Ax \quad \text{exist}\} \end{cases}$$

Theorem 1.2.2.2, Well-posedness for evolutionary equations : ([4], Chapter 2, Corollary 6.8) For a closed operator $A : D(A) \subset X \to X$, the associated (ACP) is well-posed if and only if A generates a C_0-semigroup on X

1.2.3 Semigroups, sesquilinear forms, and operators:

Definitions 1.2.3.1 On sesquilinear forms: ([9],Chapter1)

Let H be a Hilbert space over $\mathbb{C}$. Let $\langle .,. \rangle_H$ and $\|x\|_H$ denote the inner product and the corresponding norm respectively of H. An application $\mathfrak{a}$ from $\mathrm{D}(\mathfrak{a})\times\mathrm{D}(\mathfrak{a})$ into $\mathbb{C}$, where $\mathrm{D}(\mathfrak{a})$ is the domain of $\mathfrak{a}$ and a linear subspace of H;is called a sesquilinear form if for every $\alpha \in \mathbb{C}$ and $u, v, h \in H$ it satisfies the following property:

$$\mathfrak{a}(\alpha u + v, h) = \alpha\mathfrak{a}(u, h) + \mathfrak{a}(v, h) \text{ and } \mathfrak{a}(u, \alpha v + h) = \bar{\alpha}\mathfrak{a}(u, v) + \mathfrak{a}(u, h)$$

Here $\bar{\alpha}$ is the conjugate of α. if H is over $\mathbb{R}$ then the form is bilinear. We say that $\mathfrak{a}$ is **Densly Defined:** if D($\mathfrak{a}$) is dense in H. **Accretive:** if Real part of $\mathfrak{a}(u, u)$ denoted $\mathbf{R}\mathfrak{a}(u, u)$ is non-negative $\forall u \in$D($\mathfrak{a}$).**Continuous:** if there exist a non-negative constant M such that $|\mathfrak{a}(u, v)| \leq M\|u\|_{\mathfrak{a}}\|v\|_{\mathfrak{a}}$ $\forall u, v \in$ D($\mathfrak{a}$), where $\|u\|_{\mathfrak{a}}^2 = \mathbf{R}\mathfrak{a}(u, v) + \|u\|_H^2$.**Closed:**if $\|u\|_{\mathfrak{a}}$ is complete in D($\mathfrak{a}$).

Associated operator with a sesquilinear form:([9],Chapter1)
Definition 1.2.3.2 :
The operator A associated with a sesquilinear form $\mathfrak{a}$ with domain $D(\mathfrak{a})$ on a Hilbert space H is defined as follows:

$$\begin{cases} D(A) := \{u \in D(\mathfrak{a}) : \exists v \in H \text{ such that : } \mathfrak{a}(u, \phi) = \langle v, \phi\rangle_H \quad \forall\phi \in D(\mathfrak{a})\} \\ \quad Au := v \end{cases}$$

Proposition 1.2.3.3 ([9], Proposition 1.22) If a bilinear form is densely defined, accretive, continuous, and closed on a Hilbert space H, and A is the operator associated with it. Then A is densly defined, and $\forall\lambda \geq 0$, the operator $(\lambda + A)$ is invertible and it's inverse $(\lambda + A)^{-1}$ is bounded and $\|\lambda(\lambda + A)^{-1}f\| \leq \|f\| \quad$ for all $\lambda \geq 0, f \in H$
Remark: If $A =: -B$, i.e the operator A is the negative of some operator B, then substituting B in the previous proposition would result in B being dissipative.
Proposition 1.2.3.4 : ([9], Proposition 1.24) If a sesquilinear form is symmetric then the operator associated with it is self-adjoint.
Proposition 1.2.3.5 :([9], Proposition 1.51) If a bilinear form is densely defined, accretive, continuous, and closed on a Hilbert space H, and A is the operator associated with it. Then the operator $-A$ is the generator of a C_0-contraction-semigroup on H.

2

Heat Equation on non-compact Graphs

2.1 Introduction and Framework:

In the sequel, we will be considering a system of the heat equation on a non-compact graph, precisely we will define the heat equations on each edge and we will impose non-local Kirchhoff-type and continuity boundary conditions on the vertices(similar B.C were treated in [7],[8] and [5]). The aim is to prove the well-posedness of such a problem, and we shall do it in two ways as mentioned before. We first transform this system of heat equations into an abstract Cauchy problem (ACP), then for the first method, we use similar tools from "semigroups associated to sesquilinear form" as was used in ([7],[8] and [5]) for the case of compact graphs, we will proceed similarly as these papers but prove the results for the generalized case of non-compact graphs. We will show that the operator in the ACP generates a strongly-continuous semigroup C_0-SG, from which the well-posedness follows immediately. Concerning the second method, we will prove a statement that enables us to use the results in ([3]) and directly get the well-posedness without further complications.

Consider a network represented by a Non-Compact FENM Graph G, with n vertices $v_1, ..., v_n$, m edges in total, composed from k-directed edges $\underline{e}_1, ..., \underline{e}_k$, and $s := m - k$-leads $\overline{e}_{k+1}, ..., \overline{e}_m$. The edges are normalized and parameterized on the interval $[0,1]$ and the leads are parameterized on $\mathbb{R}_+$=$[0,\infty)$.Let $\Gamma(v_i)$=:$\{j \in \{1,...,m\} : e_j(0) = v_i \vee e_j(1) = v_i\}$ denotes the set of incident edge's indexes for each vertex. The structure of G is given by the previously mentioned Outgoing and Incoming Incidence Matrices denoted here by $\underline{\Phi}^+ := \left(\underline{\phi}^+_{ij}\right)_{n\times k}$, $\underline{\Phi}^- := \left(\underline{\phi}^-_{ij}\right)_{n\times k}$ and an additional outgoing incidence matrix corresponding to the incidence between the vertices and the leads, denoted by $\overline{\Phi}^+ := \left(\overline{\phi}^+_{ij}\right)_{n\times s}$. These Matrices are defined similarly as in the previous chapter:

$$\underline{\phi}^+_{ij} := \begin{cases} 1, & \text{if } \underline{e}_j(0) = v_i, \\ 0, & \text{otherwise} \end{cases} \quad ; \quad \underline{\phi}^-_{ij} := \begin{cases} 1, & \text{if } \underline{e}_j(1) = v_i, \\ 0, & \text{otherwise} \end{cases} \quad ; \quad \overline{\phi}^+_{ij} := \begin{cases} 1, & \text{if } \overline{e}_j(0) = v_i, \\ 0, & \text{otherwise} \end{cases}$$

We also consider the Incidence Matrix of G denoted and defined as : $\Phi{=}\overline{\Phi}^{+}{+}\underline{\Phi}^{+}{-}\underline{\Phi}^{-}$. This definition should be thought of as adding rows of zeros for each of the individual matrices so that the sum is well-defined.

System of equations: We consider a heat equation on each edge with the following conditions:

$$\text{(SE:)}\begin{cases} \dot{u}_j(t,x) = & \left(u_j''\right)(t,x) & t\in(0,\infty), \begin{cases} x\in(0,1) \text{ if } j\in\{1,\dots,k\} \\ x\in(0,\infty) \text{ if } j\in\{k+1,\dots,m\}\end{cases} \\ u_j\left(t,\mathrm{v}_i\right) = & u_\ell\left(t,\mathrm{v}_i\right) =: q_i(t), & t\in(0,\infty), \forall j,\ell\in\Gamma\left(\mathrm{v}_i\right), i=1,\dots,n \\ [Mq(t)]_i = & -\sum_{j=1}^{m}\phi_{ij}u_j'\left(t,\mathrm{v}_i\right), & t\in(0,\infty), i=1,\dots,n \\ u_j(0,x) = & \mathrm{u}_j(x), & \begin{cases} x\in[0,1] \text{ if } j\in\{1,\dots,k\} \\ x\in\mathbb{R}_+ \text{ if } j\in\{k+1,\dots,m\}\end{cases} \end{cases}$$

Settings and notations:
Here $u_j(t,.)$ is a function defined on the edge e_j. We denote $u_j(t,.)$ at 0 or 1 by $u_j(t,\mathrm{v}_i)$ if $e_j(1){=}\mathrm{v}_i$ or $e_j(0){=}\mathrm{v}_i$ and $u_j'(t,\mathrm{v}_i){=}0$ if $j\notin\Gamma(\mathrm{v}_i)$. A function on the whole graph $u(t,.)$ is defined by $u(t,.){=}(\underline{u}(t,.),\overline{u}(t,.))^\intercal$, where $\underline{u}(t,.) = (\underline{u_1}(t,.),...,\underline{u_k}(t,.))^\intercal \in (X[0,1])^k$ and $\overline{u}(t,.) = (\overline{u_{k+1}}(t,.),...,\overline{u_m}(t,.))^\intercal \in (X[0,\infty))^s$, where $(X[0,1])^k$ and $(X[0,\infty))^s$ are two appropriately defined (real) functional spaces over $[0,1]$ and $[0,\infty)$ respectively. Here, $M{=}(b_{ij})_{n\times n}$ is assumed to be a real, symmetric and negative semidefinite matrix. The second line in (SE), is called the continuity condition and it means that all edges adjacent to a vertex v_i must share a common value denoted by $q_i(t)$. The third line is a non-local Kirchhoff-type B.C.
Using the previously defined incidence matrices, the Kirchhoff law can be rewritten as

$$Mq(t) = -\overline{\Phi}^{+}\overline{u}'(t,0) - \underline{\Phi}^{+}\underline{u}'(t,0) + \underline{\Phi}^{-}\underline{u}'(t,1),\ t\geq 0.$$

2.2 Boundary Spaces, Operators, and the ACP

In this subsection, we will rewrite the heating system into an ACP. Because of the presence of the coordinate x along the edges, we can define naturally a Lebesgue measure on the whole graph. Therefore we can choose the previously mentioned functional spaces to be: $X[0,1] := \mathbf{L}^2(0,1)$ and $X[0,\infty) := \mathbf{L}^2(0,\infty)$, then we denote by $\underline{X} := (\mathbf{L}^2(0,1))^k$ and $\overline{X} := (\mathbf{L}^2(0,\infty))^s$ the state space of the k directed edges and s leads respectively, and let $\mathbf{X} := \underline{X}\times\overline{X}$ denote the state space of all m edges.

Proposition 1: $\mathbf{X}$ is a Hilbert space with the natural inner product:

$$\langle u,v\rangle_{\mathbf{X}} := \sum_{j=1}^{k}\int_0^1 \underline{u}_j(x)\underline{v}_j(x)dx + \sum_{j=k+1}^{m}\int_0^\infty \overline{u}_j(x)\overline{v}_j(x)dx, \quad \underline{u},\underline{v}\in\underline{X} \text{ and } \overline{u},\overline{v}\in\overline{X}$$

Proof: $\mathbf{L}^2(0,1)$ and $\mathbf{L}^2(0,\infty)$ are Hilbert spaces with their natural inner product, using **Definition 1.2.1.1** the proof follows immediately.

We need now to construct boundary operators and spaces. We start by considering first the continuity condition. Let's introduce the continuity boundary operator L defined by:

$$\begin{cases} D(L) := \left\{\ u\in(C[0,1])^k\times\left(C\left(\mathbb{R}_+\right)\right)^s : u_j\left(\mathrm{v}_i\right) = u_l\left(\mathrm{v}_i\right) \quad \forall j,l\in\Gamma\left(\mathrm{v}_i\right), i=\overline{1,n}\ \right\} \\ Lu := \left(q_1,\cdots,q_n\right)^\top = q\in\mathbb{R}^n; q_i = u_j\left(\mathrm{v}_i\right) \text{ for some } j\in\Gamma\left(\mathrm{v}_i\right), i=\overline{1,n} \end{cases}$$

Let's define now the Laplace operator A_{max} on $\mathbf{X}$ defined by :

$$\begin{cases} D\left(A_{\max}\right) := \left(H^2(0,1)\right)^k \times \left(H^2\left(0,\infty\right)\right)^s \cap D(L) \\ A_{\max} := \begin{pmatrix} \frac{\partial^2}{\partial x^2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\partial^2}{\partial x^2} \end{pmatrix}_{(m\times m)} \end{cases}$$

Consider the following operator, called **feedback operator**, defined in the following way :

$$\begin{cases} D(C) := D\left(A_{\max}\right) \\ Cu := -\overline{\Phi}^{+}\overline{u}'(0) - \underline{\Phi}^{+}\underline{u}'(0) + \underline{\Phi}^{-}\underline{u}'(1) \end{cases}$$

Using these operators and spaces we can now reformulate the system of heat equations defined on each edge, as an abstract Cauchy problem in the following way:

$$(ACP)\begin{cases} \dot{u}(t) = Au(t), t > 0 \\ u(0) = u^0 = \left(u_1^0, \ldots, u_m^0\right)^\top \end{cases} \text{ where } A \text{ is defined by: } \begin{cases} A := A_{\max} \\ D(A) := \{u \in \mathbf{X} \text{ and } MLu = Cu\} \end{cases}$$

2.3 Well-posedness of the abstract Cauchy problem:

2.3.1 First Method:

We will first define a sesquilinear form and its associated operator, prove that it satisfies useful properties that can be in some sense inherited by this operator, and these properties in turn would be sufficient for the operator to generate a C_0-semigroup. We will also show that this operator is in fact the operator A in the ACP, and well-posedness then follows immediately. Since the spaces in our settings are over the set of real numbers, then the sesquilinear form in such cases is called the bilinear form. The results proven in the sequel are generalizations to results found in (section 2 of [5]) while the technical and detailed parts for the case without leads can also be found for example in ([7]and [8]).

Consider the bilinear form $\mathfrak{a}$ defined on $\mathbf{X}$ by:

$$\begin{cases} \mathfrak{a}(u,v) = \sum_{j=1}^{k} \int_0^1 \underline{u}_j' \underline{v}_j' dx + \sum_{j=k+1}^{m} \int_0^\infty \overline{u}_j' \overline{v}_j' dx - \sum_{i,h=1}^{n} b_{ih} q_h r_i \\ D(\mathfrak{a}) := V := \left(H^1(0,1)\right)^k \times \left(H^1(\mathbb{R}_+)\right)^s \cap D(L) \\ \text{Where } Lu = q \text{ and } Lv = r \end{cases}$$

From $\mathfrak{a}$ we define its associated operator $(B, D(B))$ by :

$$\begin{cases} D(B) := \{u \in V : \exists v \in \mathbf{X} \text{ such that : } \mathfrak{a}(u,\phi) = \langle v,\phi\rangle_{\mathbf{X}} \quad \forall \phi \in V\} \\ Bu := -v \end{cases}$$

Proposition 2.3.1.1: The associated operator $(B, D(B))$ of $\mathfrak{a}$ is $(A, D(A))$ in the (ACP).
Proof: Let $u \in D(A) \implies \forall\ v \in V$ we have :

$$\mathfrak{a}(u,v) = \sum_{j=1}^{k} \int_0^1 \underline{u}_j' \underline{v}_j' dx + \sum_{j=k+1}^{m} \int_0^\infty \overline{u}_j' \overline{v}_j' dx - \sum_{i,h=1}^{n} b_{ih} q_h r_i$$

.
Consider these individual parts:

$$(a) := \sum_{j=1}^{k} \int_0^1 \underline{u}_j' \underline{v}_j' dx \quad (b) = \sum_{j=k+1}^{m} \int_0^\infty \overline{u}_j' \overline{v}_j' dx \quad (c) = \sum_{i,h=1}^{n} b_{ih} q_h r_i.$$

Then integrating by parts we have :

$$(a) = \sum_{j=1}^{k} \left[\underline{u}_j' \underline{v}_j\right]_0^1 - \sum_{j=1}^{k} \int_0^1 \underline{u}_j'' \underline{v}_j dx \quad (b) = \sum_{j=k+1}^{m} \left[\overline{u}_j' \overline{v}_j\right]_0^\infty - \sum_{j=k+1}^{m} \int_0^\infty \overline{u}_j'' \overline{v}_j dx \quad (c) = \langle Mq, r\rangle_{\mathbb{R}}.$$

We know that :

$$\underline{v}_j(0) = \sum_{i=1}^{n} \underline{\phi}_{ij}^+ r_i \quad ; \quad \underline{v}_j(1) = \sum_{i=1}^{n} \underline{\phi}_{ij}^- r_i \quad \text{and} \quad \underline{u}_j'(1) = \underline{u}_j'(0) = \underline{u}_j'(\mathrm{v}_i)$$

Then $\left[\underline{u}_j' \underline{v}_j\right]_0^1 = \underline{u}_j'(1)\underline{v}_j(1) - \underline{u}_j'(0)\underline{v}_j(0) = \sum_{i=1}^n (\underline{\phi}_{ij}^- - \underline{\phi}_{ij}^+) r_i \underline{u}_j'(\mathrm{v}_i)$
Suppose for now that $\overline{v} \in (C_0(\mathbb{R}_+))^s$ this implies that $\left[\overline{u}_j' \overline{v}_j\right]_0^\infty = -\overline{u}_j'(0)\overline{v}_j(0) = -\sum_{i=1}^n \overline{\phi}_{ij}^+ r_i \overline{u}_j'(\mathrm{v}_i)$
Hence we have :

$$\begin{cases} \sum_{j=1}^k \left[\underline{u}_j' \underline{v}_j\right]_0^1 = \sum_{i=1}^n r_i \sum_{j=1}^k (\underline{\phi}_{ij}^- - \underline{\phi}_{ij}^+) \underline{u}_j'(\mathrm{v}_i) \\ \sum_{j=k+1}^s \left[\overline{u}_j' \overline{v}_j\right]_0^\infty = -\sum_{i=1}^n r_i \sum_{j=k+1}^m \overline{\phi}_{ij}^+ \overline{u}_j'(\mathrm{v}_i) \end{cases}$$

$$\implies \sum_{j=1}^{k} \left[\underline{u}_j' \underline{v}_j\right]_0^1 + \sum_{j=k+1}^{m} \left[\overline{u}_j' \overline{v}_j\right]_0^\infty = \sum_{i=1}^{n} r_i \sum_{j=1}^{m} (\underline{\phi}_{ij}^- - \underline{\phi}_{ij}^+ - \overline{\phi}_{ij}^+) u_j'(\mathrm{v}_i)$$

From the Kirchhoff condition in matrix form we have:

$$\sum_{j=1}^{m} (\underline{\phi}_{ij}^- - \underline{\phi}_{ij}^+ - \overline{\phi}_{ij}^+) u_j'(\mathrm{v}_i) = \sum_{h=1}^{n} b_{ih} q_h$$

i.e it is equal to i-th coordinate of the vector $[Mq]$.

$$\implies \sum_{j=1}^{k} \left[\underline{u}_j' \underline{v}_j\right]_0^1 + \sum_{j=k+1}^{m} \left[\overline{u}_j' \overline{v}_j\right]_0^\infty = \sum_{i=1}^{n} \sum_{h=1}^{n} b_{ih} q_h r_i \quad \text{which is equal to (c)}$$

$$\implies (a) + (b) + (c) = -\Big[\sum_{j=1}^{k} \int_0^1 \underline{u}_j'' \underline{v}_j dx + \sum_{j=k+1}^{m} \int_0^\infty \overline{u}_j'' \overline{v}_j dx\Big]$$

$$\implies \mathfrak{a}(u, v) = -\langle Au, v\rangle_{\mathbf{X}}$$

It is now sufficient to show that if $(C_0(\mathbb{R}_+))^s$ is dense in $(H^1(0,\infty))^s$ in the $H^1 - norm$, then the previous equality holds for every $v \in V$, but This this density is true, therefore the previous implication holds.
Now we need to show the converse statement: Let $u \in D(C) \implies \exists g \in \mathbf{X}$ such that:

$$(P): \quad \mathfrak{a}(u, v) = \langle g, v\rangle_{\mathbf{X}} = \sum_{j=1}^{k} \int_0^1 \underline{g}_j \underline{v}_j dx + \sum_{j=k+1}^{k} \int_0^\infty \overline{g}_j \overline{v}_j dx$$

Let $\underline{v}_j \in H_0^1(0,1)$ and $\overline{v}_j \in H_0^1(0,\infty)$, consider the functions $[0, ..., \underline{v}_j, ..., 0]^T$ and $[0, ..., \overline{v}_j, ..., 0]^T$, Then using (P) on such functions individually we get

$$\int_0^1 \underline{u}_j' \underline{v}_j' dx = \int_0^1 \underline{g}_j \underline{v}_j dx \quad \text{for all } j = 1, ..., k, \quad \underline{v}_j \in H_0^1(0,1)$$

$$\int_0^\infty \overline{u}'_j \overline{v}'_j dx = \int_0^\infty \overline{g}_j \overline{v}_j dx \quad \text{for all } j = k+1, ..., m, \quad \overline{v}_j \in H^1_0(0,\infty)$$

By definition of the weak derivatives this means that $\underline{u}'_j \in H^1(0,1)$ for all $j = 1, ..., k$, and $\overline{u}'_j \in H^1(0,\infty)$ for all $j = k+1, ..., m,$. Then it follows that $\underline{u} \in (H^2(0,1))^k$ and $\overline{u} \in (H^2(0,\infty))^s$, therefor $u \in (H^2(0,1))^k \times (H^2(0,\infty))^s$.

Using similar integration by part from the proof of the first inclusion and applying (P) for all $v \in V_0$ we get

$$\sum_{i=1}^{n} r_i \sum_{j=1}^{m} (\underline{\phi}^-_{ij} - \underline{\phi}^+_{ij} - \overline{\phi}^+_{ij}) u'_j(\mathrm{v}_i) = \sum_{i,h=1}^{n} b_{ih} q_h r_i$$

since this holds for any v we have :

$$\sum_{j=1}^{m} (\underline{\phi}^-_{ij} - \underline{\phi}^+_{ij} - \overline{\phi}^+_{ij}) \underline{u}'_j(\mathrm{v}_i) = \sum_{h=1}^{n} b_{ih} q_h \quad \text{for all i=1,...,n,}$$

But this last formula is nothing but the Kirchhoff condition, therefore $u \in D(A)$ and we get from the integration by parts :

$$-\sum_{j=1}^{k} \int_0^1 \underline{u}''_j \underline{v}_j dx - \sum_{j=k+1}^{k} \int_0^\infty \overline{u}''_j \overline{v}_j dx = \sum_{j=1}^{k} \int_0^1 \underline{g}_j \underline{v}_j dx + \sum_{j=k+1}^{k} \int_0^\infty \overline{g}_j \overline{v}_j dx$$

Which holds for all $v \in V$. which implies that $Av = -g$.

Proposition 2.3.1.2 : $\mathfrak{a}$ is densely defined, continuous, accretive, closed, and symmetric.

Proof:

Densely defined: We have $V := (H^1(0,1))^k \times (H^1(0,\infty))^s \cap D(L)$.

We know that $(H^1(0,1))^k$,$(C([0.1]))^k$ and $(H^1(0,\infty))^s$,$(C(\mathbb{R}_+))^s$ are dense in $(\mathbf{L}^2(0,1))^k$ and $(\mathbf{L}^2(0,\infty))^s$ respectively, hence it follows that $V_1 := (H^1(0,1))^k \cap (C([0.1]))^k$ and $V_2 := (H^1(0,\infty))^s \cap (C(\mathbb{R}_+))^s$ are also dense in $(\mathbf{L}^2(0,1))^k$ and $(\mathbf{L}^2(0,\infty))^s$ respectively, this implies that $V = V_1 \times V_2$ is dense in $\mathbf{X} = (\mathbf{L}^2(0,1))^k \times (\mathbf{L}^2(0,\infty))^s$, therefore $\mathfrak{a}$ is densely defined in $\mathbf{X}$.

Accretive: By assumption of M we have : $\sum_{i,h=1}^{n} b_{ih} q_i q_h \leq 0 \implies -\sum_{i,h=1}^{n} b_{ih} q_i q_h \geq 0$

$$\implies \mathfrak{a}(u,u) \geq 0 \implies \mathfrak{a} \quad \text{is accretive}$$

Symmetric: $\mathfrak{a}$ is real-valued $\implies$ $\mathfrak{a}$ is symmetric.

Closed: we have $V := (H^1(0,1))^k \times (H^1(0,\infty))^s \cap D(L)$, denote by $H := (H^1(0,1))^k \times (H^1(0,\infty))^s$, notice that V is a Hilbert space with the natural inner product :

$$\langle u, v \rangle_H := \langle \underline{u}, \underline{v} \rangle_{(H^1(0,1))^k} + \langle \overline{u}, \overline{v} \rangle_{(H^1(\mathbb{R}_+))^s}$$

Where

$$\begin{cases} \langle \underline{u}, \underline{v} \rangle_{(H^1(0,1))^k} := \sum_{j=1}^{k} \int_0^1 (\underline{u}'_j \underline{v}'_j + \underline{u}_j \underline{v}'_j) dx \\ \langle \underline{u}, \underline{v} \rangle_{(H^1(\mathbb{R}_+))^s} := \sum_{j=k+1}^{m} \int_0^\infty (\overline{u}'_j \overline{v}'_j + \overline{u}_j \overline{v}'_j) dx \end{cases}$$

From (Lemma 3.1 in [7]) we have that $\langle \underline{u}, \underline{v} \rangle_{(H^1(0,1))^k}$ is equivalent to the following inner product:

$$\langle \underline{u}, \underline{v} \rangle_{V_1} := \sum_{j=1}^{k} \int_0^1 \underline{u}'_j \underline{v}'_j dx \quad \text{for} \quad \underline{u}, \underline{v} \in V_1.$$

Proof for this can be found in (Lemma 3.1: [7]) where they used the Poincare inequality to prove it, but this inequality also holds for domains bounded in one direction, i.e it holds for $\mathbb{R}_+$ ([6], theorem 12.17). Therefore we also have that $\langle \underline{u}, \underline{v}\rangle_{(H^1(0,\infty))^s}$ is equivalent for the inner product:

$$\langle \overline{u}, \overline{v}\rangle_{V_2} := \sum_{j=k+1}^{m} \int_0^{\infty} \overline{u}_j' \overline{v}_j' dx \quad \text{for} \quad \overline{u}, \overline{v} \in V_2.$$

Therefore $\langle u, v\rangle_H$ is equivalent to $\langle u, v\rangle_V := \langle \underline{u}, \underline{v}\rangle_{V_1} + \langle \overline{u}, \overline{v}\rangle_{V_2}$.
Now recall that the form $\mathfrak{a}$ is defined on $\mathbf{X}$ by:

$$\begin{cases} \mathfrak{a}(u,v) = \sum_{j=1}^{k} \int_0^1 \underline{u}_j' \underline{v}_j' dx + \sum_{j=k+1}^{m} \int_0^{\infty} \overline{u}_j' \overline{v}_j' dx - \sum_{i,h=1}^{n} b_{ih} q_h r_i \\ D(\mathfrak{a}) := V \end{cases}$$

$$\implies \mathfrak{a}(u,v) = \langle u, v\rangle_V - \sum_{i,h=1}^{n} b_{ih} q_h r_i$$

We need now to show that V is complete with the norm $\|.\|_{\mathfrak{a}}$. It is sufficient to show that $\|.\|_{\mathfrak{a}}$ is equivalent with $\|.\|_V$ because we know that $(V = D(\mathfrak{a}), \|.\|_V)$ is complete.
By definition we have : $\|u\|_{\mathfrak{a}}^2 := \mathfrak{a}(u,u) + \|u\|_{\mathbf{X}}^2$

$$\implies \|u\|_{\mathfrak{a}}^2 = \sum_{j=1}^{k} \int_0^1 \underline{u}_j'^2 dx + \sum_{j=k+1}^{m} \int_0^{\infty} \overline{u}_j'^2 dx - \sum_{i,h=1}^{n} b_{ih} q_h r_i + \sum_{j=1}^{k} \int_0^1 \underline{u}_j^2 dx + \sum_{j=k+1}^{m} \int_0^{\infty} \overline{u}_j^2 dx$$

$$= \sum_{j=1}^{k} \int_0^1 (\underline{u}_j'^2 + \underline{u}_j^2) dx + \sum_{j=k+1}^{m} \int_0^{\infty} (\overline{u}_j'^2 + \overline{u}_j^2) dx - \sum_{i,h=1}^{n} b_{in} q_h r_i$$

$$= \langle u, u\rangle_H - \sum_{i,h=1}^{n} b_{in} q_h r_i$$

Because $H^1(0,1)$ and $H^1(0,\infty)$ are continuously embedded in $C[0,1]$ and $C(\mathbb{R}_+)$ respectively we have:

$$|q_i| \leq \max(\max_{1\leqslant j\leqslant m} \max_{x\in[0,1]} |\underline{u}_j(x)|; \max_{1\leqslant j\leqslant m} \max_{x\in\mathbb{R}_+} |\overline{u}_j(x)|)$$

$$\implies |q_i| \leq \max_{1\leqslant j\leqslant m} \max_{x\in[0,1]} |\underline{u}_j(x)| + \max_{1\leqslant j\leqslant m} \max_{x\in\mathbb{R}_+} |\overline{u}_j(x)|)$$

$$\implies |q_i| \leq \max_{1\leqslant j\leqslant m} \|\underline{u}_j\|_{H^1(0,1)} + \max_{1\leqslant j\leqslant m} \|\overline{u}_j\|_{H^1(0,\infty)}$$

$$\implies |q_i| \leq \sum_{j=1}^{k} \|\underline{u}_j\|_{H^1(0,1)} + \sum_{j=k+1}^{m} \|\overline{u}_j\|_{H^1(0,\infty)}$$

By definition we have: $\sum_{j=1}^{k} \|\underline{u}_j\|_{H^1(0,1)} =: \|\underline{u}\|_{(H^1(0,1))^k}$ and $\sum_{j=k+1}^{m} \|\overline{u}_j\|_{H^1(0,\infty)} =: \|\overline{u}\|_{(H^1(0,\infty))^s}$

$$\implies |q_i| \leq \|u\|_H$$

but since $\|u\|_H$ is equivalent to $\|u\|_V$ this implies that $\exists N \in \mathbb{R}$ such that $|q_i| \leq N\|u\|_V$.

We have: $\|u\|_{\mathfrak{a}}^2 = \langle u, u\rangle_H - \sum_{i,h=1}^{n} b_{ih} q_h r_i \implies \|u\|_{\mathfrak{a}}^2 \leq |\langle u, u\rangle_H| + |- \sum_{i,h=1}^{n} b_{ih} q_h r_i|$

$$\leq |\langle u,u\rangle_H| + \sum_{i,h=1}^{n} |b_{ih}||q_h||r_i|$$

$$\leq |\langle u,u\rangle_H| + N^2\|u\|_V^2 \sum_{i,h=1}^{n} |b_{ih}|$$

Let $\mathbf{b} := \sum_{i,h=1}^{n} |b_{ih}|$, using Cauchy-Schwartz inequality and since the $\langle .,.\rangle_H$ is equivalent to $\langle .,.\rangle_V$, then there exist some $M \in \mathbb{R}$ such that:

$$\|u\|_{\mathfrak{a}}^2 \leq M^2\|u\|_V^2 + N^2\|u\|_V^2\mathbf{b}$$

Let $Q := (M^2 + N^2\mathbf{b})$, then :

$$\|u\|_{\mathfrak{a}}^2 \leq Q\|u\|_V^2$$

Since $-\sum_{i,h=1}^{n} b_{ih}q_h r_i$ is non-negative,

$$\implies \|u\|_{\mathfrak{a}}^2 - \langle u,u\rangle_H \geq 0$$

$$\implies \|u\|_{\mathfrak{a}}^2 \geq M^2\|u\|_V^2$$

$$\implies \|u\|_{\mathfrak{a}} \text{ is equivalent to } \|u\|_V$$

$$\implies \mathfrak{a} \text{ is closed in } \mathbf{X}$$

Continuity: Let $u, v \in V$ we have:

$$|\mathfrak{a}(u,v)| \leq |\sum_{j=1}^{k} \int_0^1 \underline{u}_j'\underline{v}_j' dx + \sum_{j=k+1}^{m} \int_0^\infty \overline{u}_j'\overline{v}_j' dx| + \sum_{i,h=1}^{n} |b_{in}||q_h||r_i|$$

$$\leq |\langle u,v\rangle_V| + N^2\|u\|_v\|v\|_V\mathbf{b}$$

$$\leq \|u\|_V\|v\|_V + N^2\|u\|_v\|v\|_V\mathbf{b}$$

$$\leq R\|u\|_V\|v\|_V$$

Where $R = (1 + N^2\mathbf{b})$ and we used the Cauchy-Schwarz inequality in the last step. Therefore the form is continuous.

Proposition 2.3.1.3: $(A, D(A))$ is densely defined, dissipative, self-adjoint and generates a C_0-semigroup of contractions $(T(t))_{t\geq 0}$ onf $\mathbf{X}$.
Proof: The properties of A follows from Proposition 1.2.3.3 and 1.2.3.4. A being the generator of a C_0-semigroup of contractions follows from Proposition 1.2.3.5

Corollary 2.3.1.4: The (ACP) is well-posed on $\mathbf{X}$.
Proof: The Proof follows from Proposition 1.2.1.3 and Theorem 1.2.2.2

2.3.2 Second Method:

The result concerning the well-posedness of the system of heat equations for more general boundary conditions as well as including the general case of the non-compact graph can be found in section 3 of [3]. The results were proved using different tools such as linking cosine families with semigroup theory. The aim now is to fit the framework of this project into the paper [3]. The setting used here are similar to the ones used in section 3 of [3], the difference is the boundary condition, so we will show that under an assumption on the graph, the non-local-Kirchhoff B.C considered in this report, is, in fact, a special case of the more general condition considered in [3].

Non-local boundary conditions : ([3], section 3.4)
$(N) : \overline{\Phi}^{+}\overline{M}^{+}\overline{u}(0) + \underline{\Phi}^{+}\underline{M}^{+}\underline{u}(0) + \underline{\Phi}^{-}\underline{M}^{-}\underline{u}(1) = \overline{\Phi}^{+}\overline{u}'(0) + \underline{\Phi}^{+}\underline{u}'(0) - \underline{\Phi}^{-}\underline{u}'(1)$
Where $\overline{M}^{+} = (\overline{m}^{+}_{ij})_{s\times s}$, $\underline{M}^{+} = (\underline{m}^{+}_{ij})_{k\times k}$ and $\underline{M}^{-} = (\underline{m}^{-}_{ij})_{k\times k}$ are real matrices.

Proposition 6:
Let's suppose that the graph is strongly connected. Then the non-local-Kirchhoff-type condition, i.e

$$(K): \quad -Mq = \overline{\Phi}^{+}\overline{u}'(0) + \underline{\Phi}^{+}\underline{u}'(0) - \underline{\Phi}^{-}\underline{u}'(1)$$

can be represented in the form of condition (N).
Proof: The right-hand side of equations (N) and (K) are the same, so it is sufficient to show that the $-Mq$ can be written as the left-hand side of (N). We want to write $-Mq$ as the left-hand side of (N), but notice that they are both vectors with n-coordinates. Every coordinate of the left-hand side of both equations is in fact some linear combinations of $(q_i)_{i\in\{1,\ldots,n\}}$. Since the graph is strongly connected, that means that the resulting internal subgraph G', by deleting all the leads is also strongly connected because the leads don't add any paths between the edges, then every vertex in G' has at least one outgoing and incoming edge. In terms of our problem, each lead is connected to some incoming or outgoing edge and they share a common vertex, and by continuity, the value at this vertex added by an arbitrary lead $\overline{e}_j$, denote such value by $\overline{u}_j(0)$ is in fact equal to $\underline{u}_l(0)$ or $\underline{u}_l(1)$ depending wether $\underline{e}_l$ is an outgoing or incoming edge (here $\underline{e}_l$ is some directed edge that share a common vertex with the lead $\overline{e}_j$) , then these values from the leads must be equal to some of the coordinates $q_1, ..., q_n$, then this means a linear combination of the $q_1, ..., q_n$ coming from the right-hand side of (N) can be represented by a linear combination coming from : $\underline{\Phi}^{+}\underline{M}^{+}\underline{u}(0) + \underline{\Phi}^{-}\underline{M}^{-}\underline{u}(1)$, up to some linear change in the coefficients. Therefore we can reduce this problem to the case where there are no leads but only directed m edges.

The Graph is strongly connected, so first, the number of edges is at least the number of vertices; second, remark that with such an assumption, every vertex has at least one outgoing edge this implies that every q_i can be represented by some of the coordinates of the vector $u(0)$, i.e for fixed indices $\underline{r}_1, ..., \underline{r}_n$, we have $(q_{r_1}, ..., q_{r_n}) = (u_{\underline{r}_1}(0), ..., u_{\underline{r}_n}(0))$, let's also permute the vector $((-b_{i1}), ..., (-b_{in}))$ by the same permutation as $(q_{r_1}, ..., q_{r_n})$, and we get the permuted vector

$((-b_{i\underline{r}_1}),...,(-b_{i\overline{r}_n}))$, then we have

$$[-Mq]_i = \sum_{h=1}^{n}(-b_{ih})q_h = \sum_{h=1}^{n}(-b_{i\underline{r}_h})q_{\underline{r}_h} = \sum_{h=1}^{n}(-b_{i\underline{r}_r})u_{\underline{r}_r}(0)$$

Let's suppose for now that every vertex has exactly one outgoing edge. This means that there is only one entry in the i-th row of $\underline{\Phi}^+$ that is equal to 1, let p be the column index for such entry, i.e $\underline{\phi}^+_{ip} = 1$. Then

$$[-Mq]_i = \sum_{h=1}^{n}(-b_{i\underline{r}_h})u_{\underline{r}_h}(0) = \sum_{h=1}^{n}\underline{\phi}^+_{ip}(-b_{i\underline{r}_h})u_{\underline{r}_h}(0)$$

$$= \sum_{h=1}^{n}\underline{\phi}^+_{ip}(-b_{i\underline{r}_h})u_{\underline{r}_h} + \underline{\phi}^+_{ip}.0.u_{\underline{r}_{n+1}}(0) + ... + \underline{\phi}^+_{ip}.0.u_{\underline{r}_m}(0)$$

$$\text{Let}: (-b_{i\underline{r}_{n+1}}) = (-b_{i\underline{r}_{n+2}}) = ... = (-b_{i\underline{r}_m}) = 0 \implies [-Mq]_i = \sum_{h=1}^{m}\underline{\phi}^+_{ip}(-b_{i\underline{r}_h})u_{\underline{r}_h}(0)$$

Set $\underline{M}^+ = [m^+_1, ..., m^+_m]$ where $m^+_j = [\underline{m}^+_{1j}, ..., \underline{m}^+_{mj}]^T \quad \forall j = 1, ..., m$
Let $m^+_p = [\underline{m}^+_{1p}, ..., \underline{m}^+_{mp}]^T := [(-b_{i\underline{r}_1}), ..., (-b_{i\underline{r}_m})]^T$ and for all j except p define $m^+_j = [0, ..., 0]^T$
Then, Since $\underline{\phi}^+_{ij} = 0 \ \forall j = 1, .., p-1, p+1, ..., m$ we have :

$$[-Mq]_i = \sum_{h=1}^{m}\underline{\phi}^+_{ip}(-b_{i\underline{r}_h})u_{\underline{r}_h}(0) = \sum_{h=1}^{m}\underline{\phi}^+_{ih}\underline{m}^+_{1h}u_{\underline{r}_1}(0) + ... + \sum_{h=1}^{m}\underline{\phi}^+_{ih}\underline{m}^+_{mh}u_{\underline{r}_m}(0)$$

$$= \left[\sum_{h=1}^{m}\underline{\phi}^+_{ih}\underline{m}^+_{1h}, ..., \sum_{h=1}^{m}\underline{\phi}^+_{ih}\underline{m}^+_{mh}\right]\left[u_{\underline{r}_1}(0), ..., u_{\underline{r}_m}(0)\right]^T = [\underline{\Phi}^+\underline{M}^+\underline{u}(0)]_i$$

$$\implies -Mq = \underline{\Phi}^+\underline{M}^+\underline{u}(0)$$

Now take $\underline{M}^- = 0$ Then we have:

$$\implies -Mq = \underline{\Phi}^+\underline{M}^+\underline{u}(0) + \underline{\Phi}^-\underline{M}^-\underline{u}(1)$$

Earlier we supposed that every vertex has exactly one outgoing edge, this is, in fact, sufficient, because if some vertex has more outgoing edges then again the equation would be equal up to some linear change in some of those coordinates that we took to be equal to 0 for example.

Bibliography

[1] Gregory Berkolaiko and Peter Kuchment, *Introduction to quantum graphs*, American Mathematical Soc., 2013.

[2] John B Conway, *A course in functional analysis*, Vol. 96, Springer, 2019.

[3] Klaus-Jochen Engel and Marjeta Kramar Fijavž, *Waves and diffusion on metric graphs with general vertex conditions*, arXiv preprint arXiv:1712.03030 (2017).

[4] Klaus-Jochen Engel, Rainer Nagel, and Simon Brendle, *One-parameter semigroups for linear evolution equations*, Vol. 194, Springer, 2000.

[5] Mihály Kovács and Eszter Sikolya, *On the stochastic allen-cahn equation on networks with multiplicative noise*, arXiv preprint arXiv:2011.11359 (2020).

[6] Giovanni Leoni, *A first course in sobolev spaces*, American Mathematical Soc., 2017.

[7] Delio Mugnolo, *Gaussian estimates for a heat equation on a network*, arXiv preprint arXiv:1005.2070 (2010).

[8] Delio Mugnolo and Silvia Romanelli, *Dynamic and generalized wentzell node conditions for network equations*, Mathematical Methods in the Applied Sciences **30** (200704), 681 –706.

[9] El-Maati Ouhabaz, *Analysis of heat equations on domains.(lms-31)*, Analysis of heat equations on domains.(lms-31), 2009.

[10] gen qi xu and Nikos Mastorakis, *Differential equations on metric graph* (201011).