Well-posedness of the Heat Equation on Non-Compact graph with non-local-Kirchhoff-type condition

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Parametrization

Consider a network represented by a Non-Compact finite metric Graph G, with n vertices $v_1, ..., v_n$, m edges in total, composed from k-directed edges $\underline{e}_1, ..., \underline{e}_k$, and $s := m - k$-leads $\overline{e}_{k+1}, ..., \overline{e}_m$.

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The edges are normalized and parameterized on the interval $[0, 1]$ and the leads are parameterized on $\mathbb{R}_+$=$[0, \infty)$.
Let $\Gamma(v_i)$=:$\{j \in \{1, ..., m\} : e_j(0) = v_i \vee e_j(1) = v_i\}$ denotes the set of incident edge's indexes for each vertex.

Structure of the graph

The structure of G is given by the Outgoing and Incoming Incidence Matrices corresponding to the directed edges, denoted here by $\underline{\Phi}^{+} := \left(\underline{\phi}^{+}_{ij}\right)_{n\times k}$, $\underline{\Phi}^{-} := \left(\underline{\phi}^{-}_{ij}\right)_{n\times k}$ and an additional outgoing incidence matrix corresponding to the leads, denoted by $\overline{\Phi}^{+} := \left(\overline{\phi}^{+}_{ij}\right)_{n\times s}$.

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These Matrices are defined as :

$$\underline{\phi}^+_{ij} := \begin{cases} 1, & \text{if } \underline{e}_j(0) = v_i, \\ 0, & \text{otherwise} \end{cases} \quad ; \quad \underline{\phi}^-_{ij} := \begin{cases} 1, & \text{if } \underline{e}_j(1) = v_i, \\ 0, & \text{otherwise} \end{cases}$$

$$\overline{\phi}^+_{ij} := \begin{cases} 1 & \text{if } \overline{e}_j(0) = v_i, \\ 0 & \text{otherwise} \end{cases}$$

System of Heat Equations

(SE):

$$\begin{cases} \dot{u}_j(t,x) = & \left(u_j''\right)(t,x) & t \in (0,T], \begin{cases} x \in (0,1) \text{ if } j \in \{1,\dots,k\} \\ x \in \mathbb{R}_+ \text{ if } j \in \{k+1,\dots,m\} \end{cases} \\ u_j\left(t,\mathrm{v}_i\right) = & u_\ell\left(t,\mathrm{v}_i\right) =: q_i(t), & t \in (0,T], \forall j,\ell \in \Gamma\left(\mathrm{v}_i\right), i = 1,\dots,n \\ [Mq(t)]_i = & -\sum_{j=1}^{m} \phi_{ij} u_j'\left(t,\mathrm{v}_i\right), & t \in (0,T], i = 1,\dots,n \\ u_j(0,x) = & \mathrm{u}_j(x), & \begin{cases} x \in (0,1) \text{ if } j \in \{1,\dots,k\} \\ x \in \mathbb{R}_+ \text{ if } j \in \{k+1,\dots,m\} \end{cases} \end{cases}$$

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A function on the whole graph $u(t,.)$ is defined by $u(t,.)=(\underline{u}(t,.),\overline{u}(t,.))^\intercal$, where $\underline{u}(t,.) = (\underline{u_1}(t,.),...,\underline{u_k}(t,.))^\intercal \in (X[0,1])^k$ and $\overline{u}(t,.) = (\overline{u}_{k+1}(t,.),...,\overline{u}_m(t,.))^\intercal \in (X[0,\infty))^s$, where $(X[0,1])^k$ and $(X[0,\infty))^s$ are two appropriately defined (real) functional spaces over $[0,1]$ and $[0,\infty)$ respectively.

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Here, $M{=}(b_{ij})_{n\times n}$ is assumed to be a real, symmetric and negative semi-definite matrix.

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Using the previously defined incidence matrices, the Kirchhoff law can be rewritten as

$$Mq(t) = -\overline{\Phi}^{+}\overline{u}'(t,0) - \underline{\Phi}^{+}\underline{u}'(t,0) + \underline{\Phi}^{-}\underline{u}'(t,1), \ \ t \geq 0.$$

State space:

Let $\underline{X} := (\mathrm{L}^2(0,1))^k$ and $\overline{X} := (\mathrm{L}^2(0,\infty))^s$ the state space of the k directed edges and s leads respectively, and let $\mathrm{X} := \underline{X} \times \overline{X}$ denote the state space of all m edges.

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Proposition 1:

X is a Hilbert space with the natural inner product:

$$\langle u, v\rangle_{\mathrm{X}} := \sum_{j=1}^{k} \int_0^1 \underline{u}_j(x)\underline{v}_j(x)dx + \sum_{j=k+1}^{m} \int_0^{\infty} \overline{u}_j(x)\overline{v}_j(x)dx, \quad \underline{u}, \underline{v} \in \underline{X} \text{ and } \overline{u}, \overline{v} \in \overline{X}$$

Boundary spaces and operators:

$$\begin{cases} D(L) := \left\{ \ u \in (C[0,1])^k \times (C(\mathbb{R}_+))^s : u_j(\mathrm{v}_i) = u_l(\mathrm{v}_i) \quad \forall j, l \in \Gamma(\mathrm{v}_i), i = \overline{1,n} \ \right\} \\ \\ Lu := (q_1, \cdots, q_n)^\top = q \in \mathbb{R}^n ; q_i = u_j(\mathrm{v}_i) \ \text{ for some } j \in \Gamma(\mathrm{v}_i), i = \overline{1,n} \end{cases}$$

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Let's define now the Laplace operator A_{max} on X defined by :

$$\begin{cases} D(A_{\max}) := \left(H^2(0,1)\right)^k \times \left(H^2(0,\infty)\right)^s \cap D(L) \\ \\ A_{\max} := \begin{pmatrix} \frac{\partial^2}{\partial x^2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\partial^2}{\partial x^2} \end{pmatrix}_{(m \times m)} \end{cases}$$

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Consider the following operator, called **feedback operator**, defined in the following way :

$$\begin{cases} D(C) := D(A_{\max}) \\ \\ Cu := -\overline{\Phi}^{+} \overline{u}'(0) - \underline{\Phi}^{+} \underline{u}'(0) + \underline{\Phi}^{-} \underline{u}'(1) \end{cases}$$

Abstract Cauchy problem

Using these operators and spaces we can now reformulate the system of heat equations defined on each edge, as an abstract Cauchy problem in the following way:

$$(ACP)\begin{cases} \dot{u}(t) = Au(t), t > 0 \\ u(0) = u^0 = \left(u_1^0, \dots, u_m^0\right)^\top \end{cases} \quad \text{where} \quad \begin{cases} A := A_{\max} \\ D(A) := \{u \in \mathrm{X} \text{ and } MLu = Cu\} \end{cases}$$

Bilinear form and associated operator:

Consider the bilinear form $\mathfrak{a}$ defined on X by:

$$\begin{cases} \mathfrak{a}(u,v) = \sum_{j=1}^{k} \int_0^1 \underline{u}_j' \underline{v}_j' dx + \sum_{j=k+1}^{m} \int_0^\infty \overline{u}_j' \overline{v}_j' dx - \sum_{i,h=1}^{n} b_{ih} q_h r_i \\ D(\mathfrak{a}) := V := \left(H^1(0,1)\right)^k \times \left(H^1(\mathbb{R}_+)\right)^s \cap D(L) \\ \text{Where } Lu = q \text{ and } Lv = r \end{cases}$$

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From $\mathfrak{a}$ we define its associated operator $(B, D(B))$ by :

$$\begin{cases} D(B) := \{u \in V : \exists v \in X \text{ such that : } \mathfrak{a}(u,\phi) = \langle v, \phi \rangle_X \quad \forall \phi \in V\} \\ Bu := -v \end{cases}$$

Generalized results

Proposition 2: The associated operator $(B, D(B))$ of $\mathfrak{a}$ is $(A, D(A))$ in the (ACP).

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$$\mathfrak{a}(u,v) = \sum_{j=1}^{k} \int_0^1 \underline{u}_j' \underline{v}_j' dx + \sum_{j=k+1}^{m} \int_0^{\infty} \overline{u}_j' \overline{v}_j' dx - \sum_{i,h=1}^{n} b_{ih} q_h r_i$$

. Using integration by part and the fact that $(C_0(\mathbb{R}_+))^s$ is dense in $(H^1(0,\infty))^s$ in the $H^1 - norm$ we get

$$\implies \mathfrak{a}(u,v) = -\Big[\sum_{j=1}^{k} \int_0^1 \underline{u}_j'' \underline{v}_j dx + \sum_{j=k+1}^{m} \int_0^{\infty} \overline{u}_j'' \overline{v}_j dx\Big]$$

$$\implies \mathfrak{a}(u,v) = -\langle Au, v\rangle_X$$

Proposition 3: $\mathfrak{a}$ is densely defined, continuous, accretive, closed, and symmetric.

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Proof:

We want to show that $\|u\|_{\mathfrak{a}}$ is complete in $D(\mathfrak{a})$, where
$\|u\|_{\mathfrak{a}}^2 = R\mathfrak{a}(u,v) + \|u\|_H^2$
We have that V is Hilbert with

$$\langle u, v\rangle_H := \langle \underline{u}, \underline{v}\rangle_{(H^1(0,1))^k} + \langle \overline{u}, \overline{v}\rangle_{(H^1(0,\infty))^s}$$

First step: $\langle u, v\rangle_H$ is equivalent to $\langle u, v\rangle_V := \langle \underline{u}, \underline{v}\rangle_{V_1} + \langle \overline{u}, \overline{v}\rangle_{V_2}$.,
where:

$$\langle \underline{u}, \underline{v}\rangle_{V_1} := \sum_{j=1}^{k} \int_0^1 \underline{u}_j' \underline{v}_j' dx \quad \text{for} \quad \underline{u}, \underline{v} \in V_1.$$

$$\langle \overline{u}, \overline{v}\rangle_{V_2} := \sum_{j=k+1}^{m} \int_0^\infty \overline{u}_j' \overline{v}_j' dx \quad \text{for} \quad \overline{u}, \overline{v} \in V_2.$$

Second step: $\|.\|_{\mathfrak{a}}$ is equivalent with $\|.\|_V$

Proposition 4: $(A, D(A))$ is densely defined, dissipative, self-adjoint and generates a C_0-semigroup of contractions $(T(t))_{t\geq 0}$ onf X.

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Proof:

Proposition 1.2.3.3 If a bilinear form is densely defined, accretive, continuous, and closed on a Hilbert space H, and $\mathcal{A}$ is the operator associated with it. Then $\mathcal{A}$ is densly defined, and $\forall\lambda \geq 0$, the operator $(\lambda + \mathcal{A})$ is invertible and it's inverse $(\lambda + \mathcal{A})^{-1}$ is bounded and $\|\lambda(\lambda + \mathcal{A})^{-1} f\| \leq \|f\| \quad$ for all $\lambda \geq 0, f \in H$

Proposition 1.2.3.4 : If a sesquilinear form is symmetric then the operator associated with it is self-adjoint.

Proposition 1.2.3.5 : If a bilinear form is densely defined, accretive, continuous, and closed on a Hilbert space H, and $\mathcal{A}$ is the operator associated with it. Then the operator $-\mathcal{A}$ is the generator of a C_0-contraction-semigroup on H.

Proposition 5: The (ACP) is well-posed on X.

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Proof:

Proposition 1.2.1.3 : If A is a densely defined operator then the adjoint of A is a closed operator.

Theorem 1.2.2.2, Well-posedness for evolutionary equations : For a closed operator $A : D(A) \subset X \to X$, the associated (ACP) is well-posed if and only if A generates a C_0-semigroup on X

General non local condition

$$(N): \overline{\Phi}^{+}\overline{M}^{+}\overline{u}(0) + \underline{\Phi}^{+}\underline{M}^{+}\underline{u}(0) + \underline{\Phi}^{-}\underline{M}^{-}\underline{u}(1) = \overline{\Phi}^{+}\overline{u}'(0) + \underline{\Phi}^{+}\underline{u}'(0) - \underline{\Phi}^{-}\underline{u}'(1)$$

Where $\overline{M}^{+} = (\overline{m}_{ij}^{+})_{s\times s}$, $\underline{M}^{+} = (\underline{m}_{ij}^{+})_{k\times k}$ and $\underline{M}^{-} = (\underline{m}_{ij}^{-})_{k\times k}$ are real matrices.

Statement

Proposition 6:
Let's suppose that the graph is strongly connected. Then the non-local-Kirchhoff-type condition, i.e

$$(K): \quad -Mq = \overline{\Phi}^{+}\overline{u}'(0) + \underline{\Phi}^{+}\underline{u}'(0) - \underline{\Phi}^{-}\underline{u}'(1)$$

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can be represented in the form of condition (N).

Proof: First step: the problem can be reduced to the case when we only have directed edges without any leads.
Second step :

$$[-Mq]_i = \sum_{h=1}^{n}(-b_{ih})q_h = \sum_{h=1}^{n}(-b_{i\underline{r}_h})q_{\underline{r}_h} = \sum_{h=1}^{n}(-b_{i\underline{r}_h})u_{\underline{r}_h}(0)$$

Let's suppose for now that every vertex has exactly one outgoing edge. This means that there is only one entry in the i-th row of $\underline{\Phi}^+$ that is equal to 1, let p be the column index for such entry, i.e $\underline{\phi}^+_{ip} = 1$.

Third step :

$$[-Mq]_i = \sum_{h=1}^{n} (-b_{i\underline{r}_h}) u_{\underline{r}_h}(0) = \sum_{h=1}^{n} \underline{\phi}^+_{ip} (-b_{i\underline{r}_h}) u_{\underline{r}_h}(0)$$

$$= \sum_{h=1}^{n} \underline{\phi}^+_{ip} (-b_{i\underline{r}_h}) u_{\underline{r}_h} + \underline{\phi}^+_{ip}.0.u_{\underline{r}_{n+\mathbf{1}}}(0) + ... + \underline{\phi}^+_{ip}.0.u_{\underline{r}_m}(0)$$

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$$= \sum_{h=1}^{n} \underline{\phi}^+_{ip} (-b_{i\underline{r}_h}) u_{\underline{r}_h} + \underline{\phi}^+_{ip}.0.u_{\underline{r}_{n+1}}(0) + ... + \underline{\phi}^+_{ip}.0.u_{\underline{r}_m}(0)$$

Let : $(-b_{i\underline{r}_{n+1}}) = (-b_{i\underline{r}_{n+2}}) = ... = (-b_{i\underline{r}_m}) = 0 \implies [-Mq]_i = \sum_{h=1}^{m} \underline{\phi}^+_{ip} (-b_{i\underline{r}_h}) u_{\underline{r}_h}(0)$

Set $\underline{M}^+ = [m^+_1, ..., m^+_m]$ where $m^+_j = [\underline{m}^+_{1j}, ..., \underline{m}^+_{mj}]^T \quad \forall j = 1, ..., m$
Let $m^+_p = [\underline{m}^+_{1p}, ..., \underline{m}^+_{mp}]^T := [(-b_{i\underline{r}_1}), ..., (-b_{i\underline{r}_m})]^T$ and for all j except p define $m^+_j = [0, ..., 0]^T$

Then, Since $\underline{\phi}_{ij}^{+} = 0 \ \forall j = 1, .., p-1, p+1, ..., m$ we have :

$$[-Mq]_i = \sum_{h=1}^{m} \underline{\phi}_{ip}^{+}(-b_{i\underline{r}_h})u_{\underline{r}_h}(0) = \sum_{h=1}^{m} \underline{\phi}_{ih}^{+}\underline{m}_{1h}^{+}u_{\underline{r}_1}(0) + ... + \sum_{h=1}^{m} \underline{\phi}_{ih}^{+}\underline{m}_{mh}^{+}u_{\underline{r}_m}(0)$$

$$= \left[\sum_{h=1}^{m} \underline{\phi}_{ih}^{+}\underline{m}_{1h}^{+}, ..., \sum_{h=1}^{m} \underline{\phi}_{ih}^{+}\underline{m}_{mh}^{+}\right]\left[u_{\underline{r}_1}(0), ..., u_{\underline{r}_m}(0)\right]^{T} = [\underline{\Phi}^{+}\underline{M}^{+}\underline{u}(0)]_i$$

$$\Longrightarrow \ -Mq = \underline{\Phi}^{+}\underline{M}^{+}\underline{u}(0)$$

Now take $\underline{M}^{-} = 0$ Then we have:

$$\Longrightarrow \ -Mq = \underline{\Phi}^{+}\underline{M}^{+}\underline{u}(0) + \underline{\Phi}^{-}\underline{M}^{-}\underline{u}(1)$$

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