2021/2022 Second Semester Research Project

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1 Introduction

This semester I studied two main topics: firstly, I read about singularity theory of complex analytic hypersurfaces in general, the theory of links and Milnor fibration/monodromy. In the second part of the semester I focused on singularities of normal complex surfaces, in particular their resolution graphs and links. There is a canonical way to attach a divisorial filtration to any (well behaved) normal surface singularity, from which we get a Poincar/Hilbert series. I studied certain special kinds of singularities, in which case the resolution corresponds to a combinatorial construction (a so called cyclic covering) of the embedded resolution graph.

In this summary, I mostly focus on the theory of cyclic coverings, the construction of the divisorial filtration and the Poincaré/Hilbert series.

The reason why I focus primarily on these two topics is that

1. Writing about everything would result in a summary exceeding the allowed length.
2. Why these two topics? The long-term goal would be to calculate the Poincarés series of certain divisorial filtrations. In the particular cases of interest, we have cyclic (or more generally Abelian) coverings, and understanding the combinatorial structure of the graphs behind such resolutions is essential in order to calculate the Poincaré series.

The main goal is to calulate the Poincaré series of surfaces of the form $f(x,y)+z^2=0$. While work on these calculations has begun, there are no results to show (yet, hopefully).

2 Normal Surface Singularities

This section is based on [1]. An analytic space is a space that locally behaves like an analytic set, i.e. the common zero set of analytic functions (simlarly to the notion of a scheme, which is an object that locally behaves like an affine scheme).

We can turn complex varieties into complex analytic spaces. Hence, by studying complex analytic singularities, we actually obtain information about complex varieties as well.

A normal surface, is just like for schemes, by definition an (analytic) surface, for which each local ring is an integrally closed integral domain. For any complex analytic space X we can assign a normal analytic space X_n (again, with an algebraic method similarly to that of schemes). This fact implies that normal singularities "explain" any singularities in some sense.

Definition 2.1. Let X be a normal surface, i.e. a 2-dimensional normal analytic space (or, analytic set, since we are only interested in the local behaviour). Let o be a singularity on X. The local data of o is described by the analytic set germ (X, o).

Now a local modification of the normal surface singularity (X, o) is by definition a proper analytic map $\widetilde{X} \xrightarrow{\varphi} X$ such that $\widetilde{X} \setminus \varphi^{-1}(o)$ is dense in $\widetilde{X}$, and it is an isomorphism on the complement of an analytic subset of $\widetilde{X}$ that contains $\varphi^{-1}(o)$. The preimage $\varphi^{-1}(o)$ is called the exceptional set of $\widetilde{X}$, and it is denoted E. A resolution is a modification such that $\widetilde{X}$ is a smooth analytic manifold.

Notice that in particular, modifications and resolutions are maps between analytic sets of the same dimension.

Theorem 2.2 (Zariski's Main Theorem)**.** *If (X, o) is a normal surface singularity, $\widetilde{X} \to X$ a fixed modification, then E is*

1. *Compact,*
2. *Connected,*
3. *One-dimensional.*

Now the irreducible components of the analytic curve E will be denoted $\{E_\nu\}_{\nu=1}^s$, and their genera g_ν. Now we can assign a matrix I, called the intersection matrix to the singularity o, which contains the intersection numbers $(E_\nu, E_\mu)_{\nu,\mu}$. Furthermore, let $f : (X, o) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function. Then the divisor $div(f \circ \varphi)$ on X decomposes into $div_E(f \circ \varphi) + S(f \circ \varphi)$, abbreviated as $div_E(f) + S(f)$, where $div_E(f)$ is the part supported on E, while $S(f)$ is the strict transform of the divisor of f, i.e. it is supported on the closure of $\varphi^{-1}(\{f = 0\} \setminus \{o\})$.

Proposition 2.3. *If φ is a resolution of a normal surface singularity (X, o), then the intersection matrix I assigned to o is negative definite.*

The Lattice Associated with a Resolution

Let $L = H^2(\widetilde{X}, \mathbb{Z})$. It can be proven that $\widetilde{X}$ has the same homology type as E; hence L is freely generated by the cohomology classes $\{E_\nu\}_{\nu=1}^s$. We have that L is basically a lattice $\mathbb{Z}^s$, with an intersection form I (it acts on the basis of L, hence on L). We also define $L' = H_2(\widetilde{X}, \partial\widetilde{X}, \mathbb{Z})$, it is dual to L. This relative homology group is generated by disks D_ν such that each D_ν is a transversal small disk at a generic point of E_ν. Then the duality map $L \times L' \to \mathbb{Z}$ is $(E_\nu, D_\mu) \mapsto \delta_{\nu,\mu}$, and the homological map $L \to L'$ is given by the matrix I. Since I is negative difinite, in particular nondegenerate, we have that L is embedded into L'. We further have that $H = L'/L$ satisfies $|H| = |\det(I)|$.

3 Resolution Graphs and Cyclic Coverings

This section is based on [2], [1].

Let (X, o) be a normal surface singularity, $f : (X, o) \to (\mathbb{C}, 0)$ a smooth analytic map. We say that a (strong) resolution $\phi : \widetilde{X} \to X$ is an embedded resolution, if $\phi^{-1}(V(f))$ is a normal crossing divisor in $\widetilde{X}$. Here $V(f)$, as usual, is the preimage of 0 by f. Under "normal crossing divisor" we mean that the components of the reduced strict transform divisor intersect transversally. We denote the components of E with $\{E_\nu\}$, and the components of the strict transform divisor $S(f)$ with $\{S_a\}$.

We can assign two graphs to the above setting: the embedded resolution graph, and the dual embedded resolution graph.

Embedded Resolution Graphs

$\Gamma(X, f)$ is a graph, constructed as follows. Its vertex set $W = V \coprod A$ consists of non-arrowhead vertices V, corresponding to E_ν, and arrowhead vertices A that correspond to the irreducible components S_a. The edges of $\Gamma(X, f)$ are given by the number of intersection points of two irreducible components (either E_ν or S_a).

We also decorate the graph Γ. We assign two labels to any vertex $v \in V$: first the self-intersection number e_ν (the νth diagonal element of I),and the genus g_ν (the label is written as $[g]$). The vertices $w \in W$ are all labeled with (m_w), where m_w is the multiplicity of f on the irreducible curve corresponding to w.

Similarly to the above construction, we can take the embedded resolution graph $\Gamma(X)$. It is just the graph $\Gamma(X, f)$ of any smooth f, with the arrowhead verices A deleted from the graph, and the labels (m_w) removed. Clearly, the embedded resolution graph only contains information about the resolution ϕ, and is independent of the choice of f.

Cyclic Coverings

Consider a normal surface singularity (X, o) and a germ of an analytic function $f : (X, o) \to (\mathbb{C}, 0)$. Let $X_{f,N}$ be the (normalized) cyclic N–covering of $(X, 0)$ branched along $(\{f = 0\}, o)$, defined as follows: it is the normalization $X_{f,N}$ of he fiber product $(X, o) \times_{f,b_N} (\mathbb{C}, 0)$, where $b_N : z \mapsto z^N$. It can be shown that there is a covering $X_{f,N} \to X$ that is branched along $f = 0$.

Our goal is to recover the resolution graph $\Gamma(X_{f,N})$ from $\Gamma(X, f)$, and N. In general, this is not possible, but the difference is only in "global data" of the resolution. In particular, the map induced by the Milnor fibration, arg_* is not determined by the resolution graph $\Gamma(X, f)$, and in many situations, this is the only "difference" in the information contained in these two graphs.

Cyclic Coverings of Graphs

We say that there is a $\mathbb{Z}$-action on a graph G if $\mathbb{Z}$ acts on both the vertex-set and the edge-set of G, such that the endpoints of e under the action of $g \in G$

correspond to the vertices obtained by acting g on the endpoints of e.

Definition 3.1. We call a $\mathbb{Z}$-equivariant morphism $G \to \Gamma$ a cyclic covering, if $\mathbb{Z}$ acts trivially on Γ, and the restriction of the $\mathbb{Z}$-action on G restricted to the preimage of a vertex or edge is transitive. Two cyclic coverings $p_i : G_i \to \Gamma$ are called equivalent, if there exists a $\mathbb{Z}$-equivariant isomorphism $q : G_1 \to G_2$, such that $p_2 q = p_1$.

Now for any edge $e = \{u, v\}$ of Γ, we have a "standard block" above e in G. It consists of n_1 vertices over u (denoted P_i^1) and n_2 vertices over v (denoted P_j^2), with $d[n_1, n_2]$ edges between these vertices, where $[a, b]$ is the least common multiplier of a, b. The edges are denoted e_l, $l \in \{1, \ldots d[n_1, n_2]$. The endpoints of e_k are $P_{i(k)}^1$ and $P_{j(k)}^2$ with $i(k) \equiv k\ (mod n_1)$ and $j(k) \equiv k\ (mod n_2)$.

In particular, P_i^1 and P_j^2 are connected by exactly d edges. The group action on such a standard block is given by the natural action of $\mathbb{Z}/d[n_1, n_2]\mathbb{Z}$ on the edges.

Definition 3.2. The data $(n, d) = \{\{n_v\}_{v \in V}, \{d_e\}_{e \in E}\}$ is called the covering data of the cyclic covering.

The equivalence classes of coverings that correspond to the covering data (n, d) are denoted $\mathcal{G}(n, d)$. It can be shown that $\mathcal{G}(n, d)$ is a group, and that it does not depend on d.

The Milnor Fibration

If (X, o) is a normal surface singularity embedded into $\mathbb{C}^N, 0$. Then for sufficiently small $\epsilon_0 > 0$, the spheres $S_\epsilon = \{z \in \mathbb{C}^N \mid ||z|| = \epsilon\}$ for $\epsilon < \epsilon_0$ intersect (X, x) transversally, and the differential manifold $L_X = S_\cap X$ does not depend on the choice of ϵ. The manifold L_X is called the link of (X, x). Now if we take an analytic germ $f : (X, x) \to (\mathbb{C}, 0)$, we can define the link of the germ f, denoted L_f similarly; by taking $S_\epsilon \cap f^{-1}(0)$. Again, the definition does not depend on the choice of ϵ. Note that L_X, L_f are topological in nature, and completely determined by the graphs $\Gamma(X), \Gamma(X, f)$ respectively (by a so called plumbing construction).

If we fix f as above, we have the following map: $arg = \frac{f}{|f|} : L_X \setminus L_f \to S^1$. This map can be shown to be a $\mathcal{C}^\infty$-fibration of manifolds. arg is called the Milnor fibration of the germ f. This in turn induces a map $arg_* : H_1(L_X \setminus L_f) \to \mathbb{Z}$.

Proposition 3.3. *The map arg_* completely determines the Milnor fibration.*

The Universal Covering Graph

Fix an embedded resolution with respect of a (fixed) germ f, and denote its resolution graph by $\Gamma(X, f)$. We do not give an explicit construction of the graph $G(X, f)$, instead list its properties. Note though, that the construction is toppological in nature, and depends on the existence of a geometric monodromy of the "nearby fiber" of X (a fiber close to the fiber of 0).

Proposition 3.4. *The number of connected components of the graph $G(X, f)$ equals the number of connected components of the Milnor fiber F of f.*

The above statement implies that the number of connected components $G(X, f)$ has $|\mathrm{coker}(arg_*(f))|$.

The Resolution Graphs of Cyclic Coverings

We have the N-fold covering $X_{f,N} \to X$, with a natural Galois-action $(x, z) \mapsto (x, \xi_k z)$, where ξ_k is an Nth root of unity corresponding to k, when we identify $\mathbb{Z}_N$ with the group of Nth roots of unity. If we have a point $P \in X \setminus f^{-1}(0)$ in the complement of the branch locus, then its preimage consists of exactly N points, which are cyclically permuted by the Galois action of $\mathbb{Z}_N$. Note that we have a monodromy representation of the (regular) covering $X_{f,N} \setminus \{z = 0\} \to X \setminus \{f = 0\}$: this representation is a map $\varphi_N : \pi_1(X \setminus \{f = 0\}) \to \mathbb{Z}_N$. In our particular case, this representation does not depend on the choice of basepoint.

Proposition 3.5. *If $\pi_1(arg) : \pi_1(L_X \setminus L_f) \to \pi_1(S^1)$ is the map induced by the Milnor fibration on the fundamental groups, and $mod_N : \mathbb{Z} \to \mathbb{Z}_N$ is the natural projection map, we have that*

$$\varphi_N = mod_N \circ \pi_1(arg).$$

Proposition 3.6. *If the Milnor fiber F of f has k connected components, then $X_{f,N}$ has (N, k) connected components.*

Remark 3.7. The number of singular points of $X_{f,N}$ lying above o is not determined by the resolution graph $\Gamma(X, f)$. It is, however determined by $G(X, f)$. This example shows the viability of the constuction of G.

Definition 3.8. The resolution graph $\Gamma(X_{f,N})$ is defined to be the union of resolution graphs of o_i, where o_i is the ith preimage of o by the covering map $X_{f,N} \to X$. Similarly, for a germ $z : (X_{f,N}, \{o_1, \dots o_{(N,k)}\}) \to (\mathbb{C}, 0)$ the embedded resolution graph of $(X_{f,N}, z)$ is defined to be the union of the embedded resolution graphs of z at each o_i.

Now the main theorem we have is the following:

Theorem 3.9. *Suppose that ϕ is en embedded resolution of $(f^{-1}(0), o) \subseteq (X, o)$ and $p : G(X, f) \to \Gamma(X, f)$ is the universal covering of the graph G associated with ϕ. Then the graphs $\Gamma(X_{f,N}, z)$ and $\Gamma(X_{f,N})$ can be determined from p and the integer N.*

For the constructions see [2], page 26.

4 Multigradings and Poincaré Series

Definition 4.1. If R is any ring, A an Abelian group. We say that R is A-multigraded, if there is an Abelian-group isomorphism $R \simeq_{\mathrm{Ab}} \bigoplus_{a \in A} R_a$ such that $R_a \times R_b \subseteq R_{a+b}$

We consider A-multigraded $\mathbb{C}$-algebras, where the R_a are all $\mathbb{C}$-vector spaces, $R_0 = \mathbb{C}$ and $\dim R_a < \infty$.

Definition 4.2. For an A-multigraded R ring, an M R-module is A-multigraded, if $M \simeq_{\mathrm{Ab}} \bigoplus_{a \in A} M_a$ such that $R_a M_b \subseteq M_{a+b}$.

We make the assumption that A is embedded in a $\mathbb{Z}^s$-lattice (or, at least we have an Abelian group homomorphism from A to $\mathbb{Z}^s$). Now for an A-multigraded R-module, where each M_a is a $\mathbb{C}$-vector space (note that R is a $\mathbb{C}$-algebra), we can define a Poincaré series associated to M:

$$P(M, \mathbb{t}) = \sum_{a \in A} \dim(M_a) \cdot \mathbb{t}^a \in \mathbb{Z}[[A]]$$

where $\mathbb{t}$ is an element of $\mathbb{C}^s$; hence $\mathbb{t}^a$ can be evaluated at a point $p = (p_1, \dots p_s)$ as $p_1^{a_1} \cdot \dots \cdot p_s^{a_s}$, where a_i is the ith component of the image of a by the embedding of A into $\mathbb{Z}^s$.

We will look at a particular filtration for which the Poincaré series can be defined as above. Note that similar Poincaré series can be defined more generally.

Divisorial Filtration & Poincaré Series

If we have a multigrading $M \simeq_{\mathrm{Ab}} \bigoplus_{a \in A} M_a$ of the module M, we can associate a filtration to it: $F(a) = \bigoplus_{a' \geq a} M_{a'}$, which satisfies $F(a) \subseteq F(a')$ for any $a \geq a'$. The Hilbert-series associated to this filtration is $H(\mathbb{t}) = \sum_{a \in \mathbb{Z}^s} \dim(M/F(a)) \mathbb{t}^a$. We have that $P(\mathbb{t}) = -H(\mathbb{t}) \prod_{i=1}^{s} (1 - t_i)^{-1}$.

There may be cases where we do have a filtration but we do not have a multigrading present. If the modules (vector spaces) $M/F(a)$ are finite-dimensional, a Hilbert series H can be assigned to the filtration. Then the Poincaré series P can be defined as above from H.

Now suppose that we have an universal cyclic covering (more generally, the same construction works for Abelian coverings). Then we have the following diagram:

$$\begin{tikzcd}
\widetilde{Z} \arrow[r, "r"] \arrow[rr, bend left, "\phi_c"] \arrow[dr, bend right, "p"'] & Z \arrow[r, "\psi_c"] \arrow[d, "\widetilde{c}"] & (X_c, o) \arrow[d, "c"] \\
 & (\widetilde{X}, E) \arrow[r, "\phi"] & (X, o)
\end{tikzcd}$$

where r is a resolution, X_c is the universal cyclic covering. Note that the curved arrows are defined from the straight arrows in a way that makes the above commutative (and not inherently obtained).

We have seen that for any resolution, the L group is a free abelian group generated by the irreducible components of E.

We have a so called valuation on the local ring $\mathcal{O}_{X_c,o}$, which in turn gives us a filtration. The associated Poincaré series are the main points of interest. The valuation is a function $v : \mathcal{O}_{X_c,o} \to \mathbb{Z} \cup \{\infty\}$, that satisfies the following:

1. $v(f_1 + f_2) \geq \min(v(f_1), v(f_2))$.
2. $v(f_1 \cdot f_2) = v(f_1) + v(f_2)$.

Now if we have a collection of order functions $\{v_1, \ldots v_s\}$, we obtain a $\mathbb{Z}^s$-filtation by setting $F(a) = \{f \in \mathcal{O}_{X_c,o} \mid v_i(f) \geq a_i \text{ for all } i\ \}$. (Note: this filtration also satisfies $F(a) \cap F(b) = F(\max(a, b))$, with the maximum taken coordinate-wise).

The divisorial filtration is an L' filtration of the local ring of holomorphic functions $\mathcal{O}_{X_c,o}$. For any $l' \in L'$ we set $\mathcal{F}(l') = \{f \in \mathcal{O}_{X_c,o} \mid \operatorname{div}(f \circ \phi_c) \geq p^*(l')\}$. We could obtain a Hilbert series from this filtration, but it would contain "redundant" information about the space. We have an action of $H = L'/L$ on $(X_c, 0)$, which induces an action on $\mathcal{O}_{X_c,o}$. This action keeps $F(l')$ invariant. Therefore H acts on $\mathcal{O}_{X_c,o}/F(l')$ as well. Now if we denote the Pontryagin duality of L' by θ. By the action defined above, we have a $\theta(l')$ eigenspaces in $\mathcal{O}_{X_c,o}/F(l')$. The dimension of these spaces is denoted $h(l')$.

Now the Hilbert series of the divisorial filtration is defined as $H(\mathbb{t}) = \sum_{l' \in L'} h(l')\mathbb{t}^{l'}$. The Poincaré series, as mentioned above, is obtained from H by We have that $P(\mathbb{t}) = -H(\mathbb{t}) \prod_{i=1}^{s}(1 - t_i)^{-1}$.

Note that in general, the term $\prod_{i=1}^{s}(1 - t_i)^{-1}$ is not invertible, which means that the Hilbert series contains more information than the Poincaré series.

As mentioned in the introduction, an example where the series H is not explicitly known to be determined by the graph $\Gamma(X, f)$, is the case of surfaces of the form $g(x, y) + z^2 = 0$.

References

[1] András Némethi. Normal surface singularities.

[2] András Némethi. Resolution graphs of some surface singularities, i. *Singularities in algebraic and analytic geometry*, 266:89, 2000.